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THE UNIVERSITY OF ALBERTA

A STUDY OF VISCO-PLASTICITY

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A STUDY OF VISCO-PLASTICITY" submitted by Henry Luming in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The minimum principle for velocity fields and the maximum principle for stress fields for a Bingham solid are extended to more applicable forms to obtain upper and lower bounds for the rate of work done by the unknown surface tractions. These extensions are applied to the problem of pure torsion of a square bar and close bounds are obtained for the torque over a range of material properties. In addition, other approximate solutions are presented. The torques obtained from these solutions are compared with the upper and lower bound torques and are found to be in satisfactory agreement with the bounds.

An experimental investigation of the pure torsion of circular cylinders of commercially pure lead is presented. The results obtained show that commercially pure lead behaves according to the Bingham visco-plastic model for a certain range of rate of twist per unit length, and beyond a critical region, the torque is independent of rate of twist per unit length up to the maximum rate used in the experiments. The maximum strain rate produced in the experiments was 0.36 per sec. A brief investigation of the shortening of pure lead circular cylinders during torsion is also presented and the results are discussed.

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NOMENCLATURE

a = semi-width of square cross-section in the torsion problem

e_{ij} = strain rate tensor

e_{xz}, e_{yz} = shear strain rate in the torsion problem (*)+

$\bar{e}_{xz}, \bar{e}_{yz}$ = $\frac{e_{xz}}{\dot{\theta}a}$, $\frac{e_{yz}}{\dot{\theta}a}$ dimensionless shear strain rate in the torsion problem

F_i = body forces per unit volume

I = $\sqrt{2e_{kl}e_{kl}}$ a strain rate invariant (*)

$\bar{I}$ = $\frac{I}{\dot{\theta}a}$ dimensionless strain rate invariant in the torsion problem

J = $\sqrt{\frac{1}{2}s_{kl}s_{kl}}$ a stress invariant

k = yield stress in shear

ℓ = length of the bar under torsion (*)

$\bar{\ell}$ = $\frac{\ell}{a}$ dimensionless length of the bar under torsion

n_x, n_y = x and y components of the normal vector

s_{ij} = deviatoric stress tensor

s_{xz}, s_{yz} = shear stresses in the torsion problem (*)

$\bar{s}_{xz}, \bar{s}_{yz}$ = $\frac{s_{xz}}{k}$, $\frac{s_{yz}}{k}$ dimensionless shear stresses in the torsion problem

S = surface of volume V (*)

$\bar{S}$ = $\frac{S}{a^2}$ dimensionless surface in the torsion problem

S_T = the portion of S on which surface tractions are prescribed

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+ An asterisk at the end of a description of symbol means that, in Sections 4.4 to 4.7, the symbol is used in place of its barred symbol unless specially noted.

S_V = the portion of S on which velocity is prescribed

T = torque in the torsion problem (*)

$\bar{T}$ = $\frac{T}{a^3k}$ dimensionless torque in the torsion problem

T_i = components of a stress vector applied on a surface

u, v = velocity components in the x and y directions respectively

U = stress function in the torsion problem (*)

$\bar{U}$ = $\frac{U}{ka}$ dimensionless stress function in the torsion problem

v_i = components of a velocity vector

V = volume

$\bar{V}$ = $\frac{V}{a^3}$ dimensionless volume in the torsion problem

w = warping velocity in the torsion problem or velocity component in the z direction (*)

$\bar{w}$ = $\frac{w}{\dot{\theta}a^2}$ dimensionless warping velocity in the torsion problem

x, y, z = coordinates in a rectangular coordinate system (*)

$\bar{x}$, $\bar{y}$, $\bar{z}$ = $\frac{x}{a}$, $\frac{y}{a}$, $\frac{z}{a}$ dimensionless coordinates in the torsion problem

Y = yield stress

ϵ = $\frac{\eta\dot{\theta}a}{k}$ a dimensionless parameter in the torsion problem

γ = parameter used in the Ritz's method

η = coefficient of viscosity

η_L = coefficient of viscosity in the linearized constitutive relationship

θ = angle of twist in the torsion problem

$\dot{\theta}$ = rate of twist in the torsion problem

μ = modulus of rigidity

σ_{ij} = stress tensor

τ = $\sqrt{s_{xz}^2 + s_{yx}^2}$ magnitude of the shear stress vector in the torsion problem

ω = speed of twist in rpm

NOTE: For the equations which are in tensor notations, the summation convention is implied.

CHAPTER I

INTRODUCTION

Many solids do not behave elastically under load. Even if a solid is elastic, it is so only under a certain range of load. The deviation from elastic behavior in a solid is generally termed "inelasticity". Inelasticity in a solid is caused by dissipation of mechanical energy by the solid in three ways. These three kinds of inelasticity are: (a) anelastic deformation - caused by the dissipation of mechanical energy through the interaction with flux of non-mechanical energy such as thermal, electrical, and magnetic energy; (b) viscous response - caused by the dissipation of mechanical energy through the viscous interaction between elements of the solid; (c) general plastic response - caused by the dissipation of mechanical energy through solid friction due to slip. (1)*

In a "real" solid, all three forms of inelasticity are present to some extent under various conditions. This thesis is concerned with a solid for which the second and third kinds of inelasticity mentioned above dominate. This solid is known as a "visco-plastic" solid or a Bingham solid.

In processes such as hot extrusion, hot forging, and

*Numbers without decimal enclosed by brackets designate the references listed in the Bibliography.

hot rolling, the strain rate dependent properties of the metal usually influence the load required to produce deformation. The extrusion of pure lead at room temperature is an example of such a process. One method of considering the rate effects is to use the Bingham solid model to represent the metal and the purpose of the experimental part of this thesis is to investigate how closely the behavior of commercially pure lead under torsion at room temperature approximates to that of the ideal Bingham solid. Preliminary research studies of the rate effects are usually done with pure lead, because the apparatus for performing experiments with hot metal is quite elaborate.

In the theoretical part of this thesis, extensions of the extremum principles for a Bingham solid, some approximate methods of solution of the problem of pure torsion of a visco-plastic prismatic bar with a simply-connected cross section, and some general theories of plasticity derived for the Bingham solid are presented.

It is hoped that the investigation of some methods of analysis for the problem of pure torsion will give some indications of possible methods of analysis for other visco-plastic boundary value problems, and the experimental investigation of pure torsion will lead to a possible method of determining viscosity coefficients of visco-plastic materials.

CHAPTER II

GENERAL THEORY OF VISCO-PLASTICITY

2.1 Brief Historical Background

While the development of the theory of plasticity has progressed rapidly during the last twenty years, the interest in visco-plasticity has grown only in the past few years. In earlier years, visco-plasticity was of interest mainly to rheologists, but in recent years, visco-plasticity has also been developed as a branch of applied mechanics that is concerned with a solid that possesses viscous properties when in the plastic state.

As early as 1922, a model for a visco-plastic solid subjected to simple shear was proposed by Bingham (2). For this model the shear stress due to a certain shear strain rate is the sum of the shear stress for a Newtonian fluid and the shear stress for a plastic solid due to the same shear strain rate. A solid that behaves, in simple shear, like Bingham's model is now known as a Bingham solid. In 1934, Grimshaw (3) attempted to consider the rate effect on the plastic deformation of a right circular cylinder of soft metal under axial compression. Grimshaw assumed the stress to be the sum of the stress in a Newtonian fluid, proportional to strain rate, and the stress from an elastic-type relation, proportional to

strain. The modern theory of plasticity shows that the use of an elastic-type or deformation type of relation for plastic deformation is not, in general, correct.

Apart from Grimshaw's work, practically all other papers on visco-plasticity have been based on the generalization of the model proposed by Bingham. A constitutive relation for the Bingham solid was formulated by Hohenemser and Prager in 1932 (4). The Mises yield criterion is used in this model; consequently, the constitutive relation is non-linear. Prager in 1961 (5) presented a theory of visco-plasticity based on a piecewise linear yield criterion.

In 1946 and 1947, Oldroyd (6) formulated equations for the plastic flow of a Bingham solid and solved some two-dimensional boundary layer and rectilinear flow problems, though mainly from the rheological point of view.

The first general results for a visco-plastic solid were obtained by Prager in 1954 (7). Prager derived extremum principles analogous to the principles of minimum potential energy and minimum complementary energy in classical elasticity and plasticity. Extensions of these extremum principles to more applicable forms are presented later in this thesis.

Very few problems have been solved for the visco-plastic model. Carlson (8) considered the axial compression

of a disk without accounting for inertia effect and using a piece-wise linear yield criterion. Haddow (9) solved the same problem with inertia effect considered and using the Mises yield criterion. Appleby and Prager (10) applied the linearized constitutive relation to solve the flow problem of a thin circular plate subjected to a uniformly distributed transverse load. Appleby (11) also considered the visco-plastic expansion of a long thick walled tube subjected to internal pressure.

Ziegler (12) proposed four models for dissipative media. Two of the models may be considered as visco-plastic, one a visco-plastic solid and the other a visco-plastic liquid according to the classical definitions of a solid and a liquid. Ziegler solved a simple flow problem for the dissipative media and obtained two limit theorems.

2.2 The Bingham Solid

The Bingham solid is so far the most common visco-plastic material that researchers have investigated. In the modern literature, the term Bingham solid refers to a solid that behaves as follows: When the stress state does not violate the yield criterion, the solid is rigid, that is, the strain and strain rate remain zero. When the state of stress exceeds the yield stress, visco-plastic

flow with no workhardening occurs. Then the stress intensity is the sum of two parts: the stress obtained from the same strain rate applied to a rigid perfectly plastic solid and the stress obtained from the same strain rate applied to an incompressible Newtonian fluid. A mechanical model of this solid is shown in Fig. 2.1.

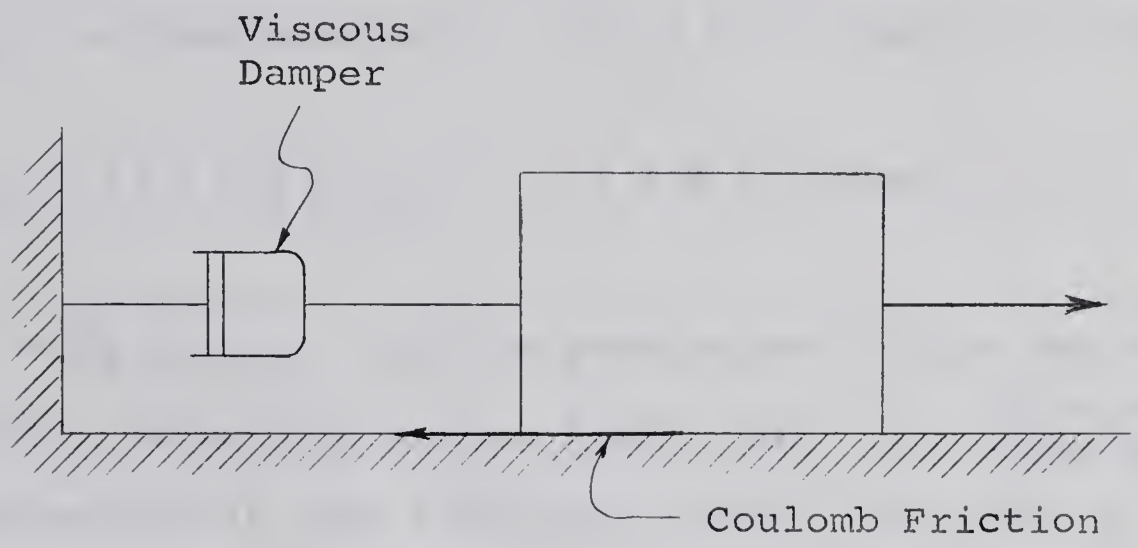

Fig. 2.1 Mechanical Model for Bingham Solid

Mathematically, the stress components are

$$s_{ij} \text{ --- Indeterminate} \qquad \text{if } e_{ij} = 0 \quad \text{that is if } f(s_{ij}) \leq k^2 \qquad (2.1)$$

$$s_{ij} = (s_{ij})_p + (s_{ij})_n \qquad \text{if } e_{ij} \neq 0 \quad \text{that is if } f(s_{ij}) > k^2, \qquad (2.2)$$

where $f(s_{ij})$ is the yield function, and $(s_{ij})_p$ and $(s_{ij})_n$ are the stress deviators for a perfectly plastic solid and a Newtonian fluid, respectively, subjected to the given strain rate e_{ij}. Using the Levy-Mises relation for rigid perfectly plastic solids (13), in which the Mises yield criterion is used,

$$(s_{ij})_p = \frac{k}{\sqrt{\frac{1}{2}e_{kl}e_{kl}}}\, e_{ij} \tag{2.3}$$

$$(s_{ij})_n = 2\eta e_{ij} \quad . \tag{2.4}$$

Therefore, the constitutive relation for Bingham solid is

$$s_{ij} \text{ --- Indeterminate} \qquad \text{if } I = 0 \quad \text{that is if } J \leqslant k \tag{2.5}$$

$$s_{ij} = 2\left[\eta + \frac{k}{I}\right] e_{ij} \qquad \text{if } I \neq 0 \quad \text{that is if } J > k \tag{2.6}$$

where $J = \sqrt{\frac{1}{2}\, s_{ij}s_{ij}}$ is the square-root of the second invariant of the deviatoric stress tensor and $I = \sqrt{2e_{kl}e_{kl}}$ is the square-root of four times the second invariant of the deviatoric strain rate tensor. To invert relation (2.6), both sides are squared. Thus,

$$J = \left[\eta + \frac{k}{I}\right] I \quad , \tag{2.7}$$

then,

$$I = \frac{J - k}{\eta} \quad . \tag{2.8}$$

After substituting (2.8) in (2.6) and rearranging, the inverted constitutive relation is

$$e_{ij} = 0 \qquad (\text{if } J < k) \tag{2.9}$$

$$e_{ij} = \frac{1}{2\eta}\left[1 - \frac{k}{J}\right] s_{ij} \qquad (\text{if } J \geqslant k) \quad . \tag{2.10}$$

2.3 Bingham Solid and Rigid Perfectly Plastic Solid

In this section, the major difference between the Bingham Solid and the rigid plastic solid is discussed. A rigid plastic solid exhibits no rate effects; the current state of stress defines the ratios of the components of the strain rate tensor but not the magnitudes. For a rigid plastic solid, the velocities and strain rates in problems in which inertial forces may be neglected can be referred to a suitable monotonically varying parameter of the problem under consideration. This is true for both perfectly plastic solids and plastic workhardening solids. The Levy-Mises relation for a rigid perfectly plastic solid may be written in the form (13)

$$s_{ij} = \frac{k}{\sqrt{\frac{1}{2}\, e_{kl} e_{kl}}}\, e_{ij} \qquad (\text{if } J = k), \tag{2.11}$$

and this clearly indicates that only the ratios of the components of the strain rate tensor are defined by the stress during plastic deformation. The constitutive relationships given by Hill (14) for a workhardening solid also indicate this.

However, if the viscosity term $2\eta e_{ij}$ is added to relation (2.11) to form the relation for Bingham solids (2.6), the magnitudes of the strain rates and not just their ratios are defined by the state of stress when $J > k$. In a visco-plastic problem, all rates must be with respect to time.

2.4 Linearization of the Constitutive Relation

The linearization of the constitutive relation of the Bingham solid by Prager (5) is summarized in this section. Prager noted that the constitutive relation in (2.9) and (2.10) can be put in the form,

$$e_{ij} = \frac{1}{2\eta} \langle F \rangle \frac{\partial F}{\partial \sigma_{ij}} \tag{2.12}$$

where
$$F = \sqrt{2}\ (J - k) = \sqrt{s_{ij} s_{ij}} - \sqrt{2}\ k \tag{2.13}$$

and
$$\langle F \rangle = \begin{cases} 0 & \text{for } F < 0 \\ F & \text{for } F \geqslant 0 \end{cases} . \tag{2.14}$$

In (2.13), the yield function, F, is a single non-linear function. The linearization involves the approximation of this function by a set of m linear functions,

$$L_{(q)} = \alpha^{(q)}_{kl}\ s_{kl} - \beta^{(q)} \tag{2.15}$$

$$(q = 1,\ 2,\ .\ .\ .\ ,\ m)$$

in which the symmetric deviators $\alpha^{(q)}_{kl}$ and the scalars $\beta^{(q)}$ are independent of s_{ij}. The number m may be chosen according to the degree of accuracy desired. With the functions $L_{(q)}$ in place of F in (2.12), and η_L for the viscosity coefficient,

$$e_{ij} = \frac{1}{2\eta_L} \sum_{q=1}^{m} \langle L_{(q)} \rangle \frac{\partial L_{(q)}}{\partial \sigma_{ij}} \quad . \tag{2.16}$$

Since $\alpha_{pp}^{(q)} = 0$, $\dfrac{\partial L_{(q)}}{\partial \sigma_{ij}} = \alpha_{ij}^{(q)}$,

then
$$e_{ij} = \frac{1}{2\eta_L} \sum_{q=1}^{m} \alpha_{ij}^{(q)} \langle L_{(q)} \rangle \ . \tag{2.17}$$

Equation (2.17) then gives the linearized constitutive relation.

The viscosity coefficients η and η_L are not, in general, the same; however, they may be made the same by multiplying the functions $L_{(q)}$ by an appropriate constant factor. For example, referring to Fig. 2.2, the Tresca yield criterion gives the following linear functions:

$$\left.\begin{array}{l}
\text{Line AB - } L_{(1)} = \sigma_1 - \sigma_2 - 2k \\
\text{Line BC - } L_{(2)} = \sigma_1 - \sigma_3 - 2k \\
\text{Line CD - } L_{(3)} = \sigma_2 - \sigma_3 - 2k \\
\text{Line DE - } L_{(4)} = \sigma_2 - \sigma_1 - 2k \\
\text{Line EF - } L_{(5)} = \sigma_3 - \sigma_1 - 2k \\
\text{Line AF - } L_{(6)} = \sigma_3 - \sigma_2 - 2k
\end{array}\right\} \tag{2.18}$$

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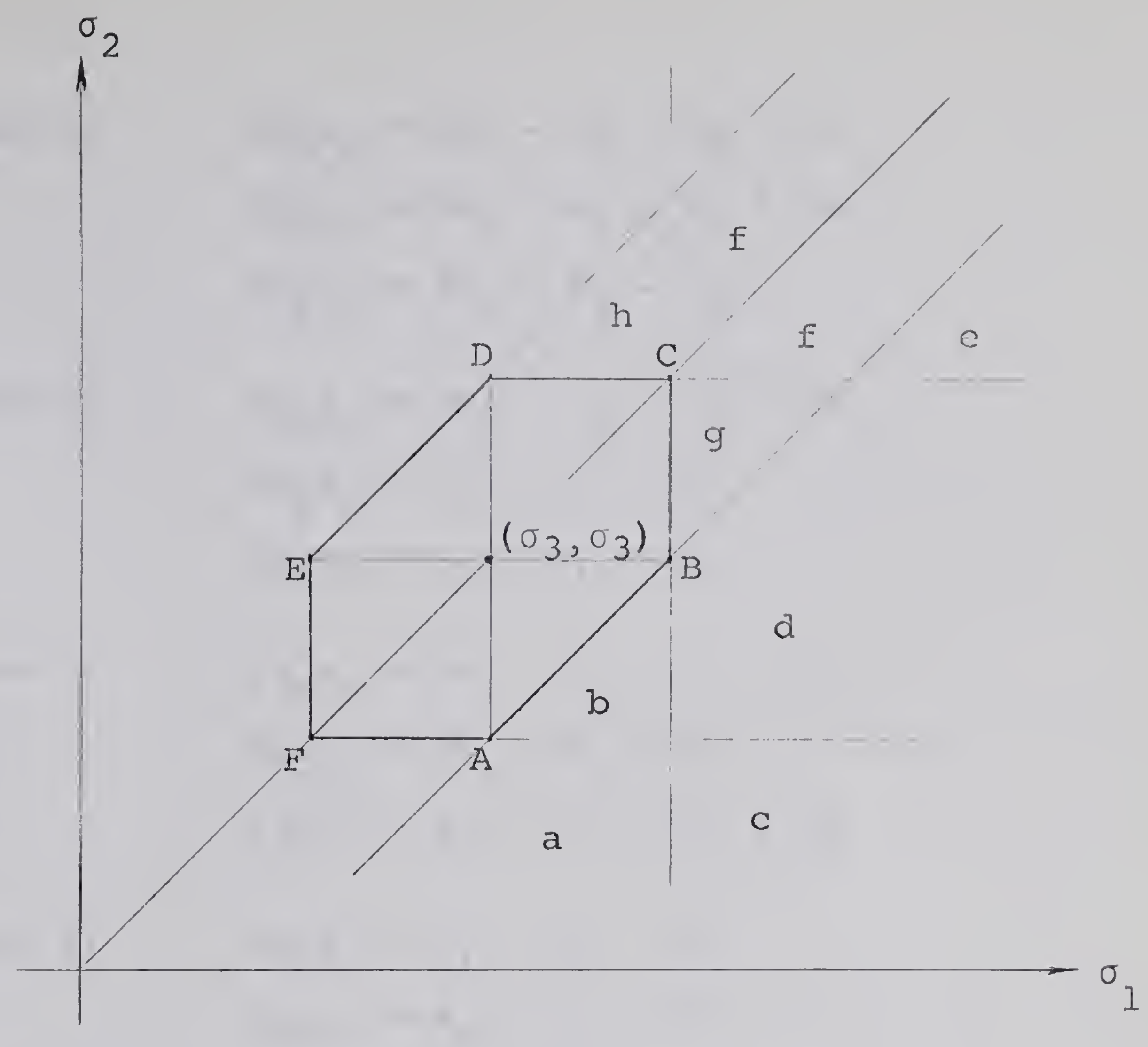

Fig. 2.2 Yield Locus in Plane σ_3 = constant for Tresca Yield Criterion

Then, from (2.16), the stress-strain rate relationships are as follows: If the stress point is in

Region a:

$$2\eta_L e_1 = \sigma_1 - \sigma_2 - 2k$$
$$2\eta_L e_2 = 2\sigma_2 - \sigma_1 - \sigma_3 + 4k$$
$$2\eta_L e_3 = \sigma_3 - \sigma_2 - 2k$$

Region b:

$$2\eta_L e_1 = \sigma_1 - \sigma_2 - 2k$$
$$2\eta_L e_2 = \sigma_2 - \sigma_1 + 2k$$
$$2\eta_L e_3 = 0$$

Region c:

$$2\eta_L e_1 = 2\sigma_1 - \sigma_2 - \sigma_3 - 4k$$
$$2\eta_L e_2 = 2\sigma_2 - \sigma_1 - \sigma_3 + 4k$$
$$2\eta_L e_3 = 2\sigma_3 - \sigma_1 - \sigma_3$$

Region d:

$$2\eta_L e_1 = 2\sigma_1 - \sigma_2 - \sigma_3 - 4k$$
$$2\eta_L e_2 = \sigma_2 - \sigma_1 + 2k$$
$$2\eta_L e_3 = \sigma_3 - \sigma_1 + 2k$$

Region e:

$$2\eta_L e_1 = 2\sigma_1 - \sigma_2 - \sigma_3 - 4k$$
$$2\eta_L e_2 = 2\sigma_2 - \sigma_1 - \sigma_3$$
$$2\eta_L e_3 = 2\sigma_3 - \sigma_1 - \sigma_2 + 4k$$

Region f:

$$2\eta_L e_1 = \sigma_1 - \sigma_3 - 2k$$
$$2\eta_L e_2 = \sigma_2 - \sigma_3 - 2k$$
$$2\eta_L e_3 = 2\sigma_3 - \sigma_1 - \sigma_2 + 4k$$

and so forth for the other regions.

Now if k approaches zero, the following relationships are obtained:

Region a:

$$2\eta_L e_2 = 2\sigma_2 - \sigma_1 - \sigma_3 = 3s_2$$

Region c:

$$2\eta_L e_1 = 2\sigma_1 - \sigma_2 - \sigma_3 = 3s_1$$
$$2\eta_L e_2 = 2\sigma_2 - \sigma_1 - \sigma_3 = 3s_2$$
$$2\eta_L e_3 = 2\sigma_3 - \sigma_1 - \sigma_2 = 3s_3$$

Region d:

$$2\eta_L e_1 = 2\sigma_1 - \sigma_2 - \sigma_3 = 3s_1$$

Region e:

$$2\eta_L e_1 = 2\sigma_1 - \sigma_2 - \sigma_3 = 3s_1$$
$$2\eta_L e_2 = 2\sigma_2 - \sigma_1 - \sigma_3 = 3s_2$$
$$2\eta_L e_3 = 2\sigma_3 - \sigma_1 - \sigma_2 = 3s_3$$

Region f:

$$2\eta_L e_3 = 2\sigma_3 - \sigma_1 - \sigma_2 = 3s_3$$
$$2\eta_L e_1 = 2\sigma_1 - \sigma_3 - \sigma_2 = 3s_1 \quad (\text{if } \sigma_1 = \sigma_2)$$
$$2\eta_L e_2 = 2\sigma_2 - \sigma_1 - \sigma_3 = 3s_2 \quad (\text{if } \sigma_1 = \sigma_2)$$

where s_1, s_2, and s_3 are the principal stress deviators. From comparing the linear relationships with the non-linear relationships having $k \to 0$, it follows that

$$\eta_L = 3\eta \quad . \tag{2.19}$$

This difference in the viscosity coefficients may be eliminated by multiplying the linear functions (2.18) by the constant factor $\frac{1}{\sqrt{3}}$.

2.5 The Boundary Value Problem

The statement of the boundary value problem is as follows: A visco-plastic material occupies the volume V and is bounded by surface S. The body force per unit volume F_i is given in V. Surface traction T_i is prescribed on surface S_T, a portion of S, and the velocity v_i is prescribed on

surface S_V , another portion of S. S_T and S_V may overlap, provided on the overlapped surface the prescribed surface tractions and velocity complement each other. For example the normal component of stress and the tangential component of velocity may be prescribed or vice-versa. The boundary value problem is then, for given F_i and the prescribed data on the surface, to determine the velocity field v_i and the stress field σ_{ij} throughout the volume V. The stress field σ_{ij} must satisfy the equilibrium equations

$$\sigma_{ij,j} + F_i = 0 \tag{2.20}$$

assuming that inertia effects may be neglected, and also it must satisfy the boundary condition

$$\sigma_{ij}n_j = T_i \qquad \text{on } S_T \ . \tag{2.21}$$

The corresponding velocity field v_i is related by the stress-strain rate relation. For the incompressible Bingham solid, the relation is

$$s_{ij} = 2\left[\eta + \frac{k}{I}\right] e_{ij} \ , \tag{2.6}$$

where
$$e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \ . \tag{2.22}$$

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For the case of incompressible solids, v_i must also satisfy the incompressibility relation,

$$e_{ii} = v_{i,i} = 0 \ . \tag{2.23}$$

Also, v_i must be equal to the prescribed value at the surface S_V. The solution of this boundary value problem was proved to be unique by Prager (7).

2.6 Extremum Principles

The utilization of variational principles to obtain solutions to physical problems has been a powerful mathematical method in classical and modern mechanics. The method depends on the formulation of extremum principles. For visco-plasticity, Prager (7) derived two extremum principles for an incompressible Bingham solid, under the boundary condition that surface tractions and velocity are prescribed on S_T and S_V, respectively, and S_V and S_T are not zero. Earlier, Ilyushin (15) has derived one of these principles assuming $S_V = 0$.

In order to facilitate the stating of the two extremum principles, two types of admissible field shall first be defined. A kinematically admissible velocity field for a visco-plastic problem is a continuous velocity field satisfying the boundary condition on S_V and the incompressibility condition. The

strain rate field derived from this velocity field furnishes a stress field. In general, this stress field will not satisfy the equilibrium equations and the stress boundary condition. A statically admissible stress field is a stress field that satisfies the equilibrium equations and the stress boundary condition on S_T. The strain rate field derived from this stress field, in general, does not satisfy the compatibility equations and cannot be derived from a velocity field.

The minimum principle for velocity fields states that among all kinematically admissible velocity fields, the actual velocity field minimizes the functional

$$H = \int_V (\eta I^2 + 2kI)\, dV - 2\int_{S_T} T_i v_i dS - 2\int_V F_i v_i dV. \quad (2.24)$$

This principle holds only for problem with $S_V \neq 0$. Functional (2.24) is similar to the potential energy in classical elasticity and plasticity; however, the first integral does not represent the total rate of energy dissipation, which is

$$\int s_{ij} e_{ij} dV = \int 2(\eta + \frac{k}{I})\, e_{ij} e_{ij} dV = \int (\eta I^2 + kI)\, dV. \quad (2.25)$$

The maximum principle for stress fields states that among all statically admissible stress fields, the actual stress field maximizes the functional

$$K = 2\int_{S_V} T_i v_i dS - \frac{1}{4\eta}\int_V \left(|J - k| + J - k\right)^2 dV . \tag{2.26}$$

There is no closed comparison between this functional and the complementary energy in classical elasticity and plasticity. This principle holds only for problem with $S_T \neq 0$. These two extremum principles can also be shown to be true when S_T and S_V overlap.

CHAPTER III

EXTENSIONS OF THE EXTREMUM PRINCIPLES FOR A BINGHAM SOLID

Upper and lower bounds for the rate of work done by the unknown surface tractions are easily obtained from the extremum principles for a rigid plastic solid. In visco-plasticity, the extremum principles stated in Section 2.6 are not readily converted to the form for obtaining upper and lower bounds for the rate of work done by the unknown surface tractions. In this chapter, extensions of the extremum principles are presented from which the bounds may be obtained.

3.1 An Extension of the Minimum Principle for Velocity Fields

Problems for which the tractions on S_T and the body force F_i are zero are considered. If v_i^* is a kinematically admissible velocity field, v_i is the solution velocity field of a problem of visco-plastic flow, and I^* and I are the corresponding invariants, then from (2.24)

$$\int_V (\eta I^{*2} + 2kI^*)\ dV \geqslant \int_V (\eta I^2 + 2kI)\ dV. \quad (3.1)$$

Since body force and prescribed surface tractions are zero and for an incompressible material $s_{ij}e_{ij} = \sigma_{ij}e_{ij}$, it follows

from (2.25) and the principle of virtual work[+] that

$$\int_{S_V} T_i v_i dS = \int_V (\eta I^2 + kI) dV . \qquad (3.2)$$

Combining (3.1) and (3.2) gives

$$\int_V (\eta I^{*2} + 2kI^*) dV \geqslant \int_V kIdV + \int_{S_V} T_i v_i dS . \qquad (3.3)$$

The term $\int_V kIdV$ represents the rate of plastic energy dissipation. Hill (14) and Prager and Hodge (13) have given the extremum principles for the rigid perfectly plastic solid and the extremum principle concerned with velocity fields states that among all kinematically admissible[++] velocity fields v_i^o the actual one minimizes

$$B^o = \int_V kI^o \, dV + \sum \int_{S_d} k \left[v^o\right] dS - \int_{S_T} T_i v_i^o \, dS \qquad (3.4)$$

where $\left[v^o\right]$ is the magnitude of velocity discontinuities in the field v_i^o and S_d are the surfaces of discontinuity.

[+] Let v_i be a continuous velocity field and σ_{ij} be an entirely unrelated stress field satisfying the equilibrium equations, further $e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ and $T_i = \sigma_{ij} n_j$, then the principle of virtual work states that

$$\int_V \sigma_{ij} e_{ij} dV = \int_S T_i v_i dS + \int_V F_i v_i dV$$

[++] A kinematically admissible velocity field for a rigid perfectly plastic solid may have velocity discontinuities.

Inequality (3.3) can be modified to give an upper bound for $\int_{S_V} T_i v_i dS$ if the velocity solution to the corresponding problem for a rigid perfectly plastic solid is known. If v_i^* is chosen as the velocity field of the rigid perfectly plastic solution and if v_i^* has no surfaces of discontinuity and v_i is the velocity field of the visco-plastic solution, it follows from (3.4) that

$$\int_V kI^* \, dV < \int_V kI dV \,, \tag{3.5}$$

since the tractions on S_T are zero.
The following inequality is obtained by combining inequalities (3.3) and (3.5).

$$\int_V (\eta I^{*2} + kI^*) dV > \int_{S_V} T_i v_i dS \tag{3.6}$$

Inequality (3.6) gives an upper bound for $\int_{S_V} T_i v_i dS$. The left hand side of (3.6) must of course be evaluated from the velocity field of a complete rigid perfectly plastic solution with no velocity discontinuities. When the viscous effect is small compared with the yield effect, that is when $\frac{\eta V}{kL} << 1$, where V is a characteristic velocity and L a characteristic length, it is reasonable to suppose that inequality (3.6) gives a close upper bound.

Many important rigid perfectly plastic solutions involve velocity discontinuities. If the velocity field v_i^+ of the corresponding complete rigid perfectly plastic solution has velocity discontinuities then it follows from (3.4) that

$$\int_V kIdV > \int_V kI^+ dV + \sum \int_{S_d} k\left[v^+\right] dS, \qquad (3.7)$$

where as before I is obtained from the unique visco-plastic velocity field. The following inequality is obtained by combining inequalities (3.3) and (3.7).

$$\int_V (\eta I^{*2} + 2kI^*)\,dV - \left[\int_V kI^+ dV + \sum \int_{S_d} k\left[v^+\right] dS\right] > \int_{S_V} T_i v_i dS, \qquad (3.8)$$

where I^* is obtained from any velocity field that is kinematically admissible for the rigid visco-plastic boundary value problem. Inequality (3.8) then gives upper bound for $\int_{S_V} T_i v_i dS$ when the corresponding rigid perfectly plastic solution has velocity discontinuities.

3.2 An Extension of the Maximum Principle for Stresses

This extension is also for problems for which the surface tractions on S_T and body force F_i are zero. If σ_{ij}^* is a statically admissible stress field, σ_{ij} is the unique stress field solution of the visco-plastic problem, and J^* and J are the corresponding stress invariants, then from (2.26)

$$2\int_{S_V} T_i v_i dS - \frac{1}{4\eta}\int_V \left[|J-k| + J-k\right]^2 dV \geqslant 2\int_{S_V} T_i^* v_i dS$$

$$- \frac{1}{4\eta} \int_V \left[|J^*-k| + J^*-k\right]^2 dV. \quad (3.9)$$

It is now required that σ_{ij}^* be chosen such that $J^* \geqslant k$. Then, it follows from (3.9) that

$$\int_S T_i v_i dS \geqslant \int_{S_V} T_i^* v_i dS - \frac{1}{2\eta}\int_V (J^*-k)^2 dV + \frac{1}{2\eta}\int_{V_d} (J-k)^2 dV, \quad (3.10)$$

where V_d is the volume of the deforming region and the equality sign is included only if the problem considered is fully deforming. From (2.8)

$$J - k = \eta I \qquad \text{if} \quad I \neq 0. \qquad (3.11)$$

Substitution of (3.11) in (3.10) gives

$$\int_{S_V} T_i v_i dS \geqslant \int_{S_V} T_i^* v_i dS - \frac{1}{2\eta}\int_V (J^*-k)^2 dV + \frac{\eta}{2}\int_{V_d} I^2 dV. \quad (3.12)$$

The integral $\int_{V_d} \eta I^2 dV$ represents the rate of viscous energy dissipation of the deforming region. From theory of slow incompressible Newtonian flow, the extremum principle for the velocity field states that among all kinematically admissible velocity fields v_i^+ the velocity solution field minimizes the functional (16)

$$K^+ = \frac{1}{2}\int_V \eta I^{+2} dV - \int_{S_T} T_i v_i^+ dS. \qquad (3.13)$$

Inequality (3.12) can be modified to give a lower bound for $\int_{S_V} T_i v_i dS$ if the velocity solution to the corresponding slow incompressible Newtonian flow is known. Let v_i^o be the velocity solution of the Newtonian flow problem; and since the velocity solution, v_i, of the visco-plastic problem is kinematically admissible for the corresponding Newtonian flow problem and the surface traction on S_T are zero, it follows from (3.13) that

$$\frac{\eta}{2}\int_V I^2 dV > \frac{\eta}{2}\int_V I^{o2} dV, \tag{3.14}$$

where I^o is obtained from v_i^o. In the integral on the left hand side, the integration over the non-deforming region is zero. Therefore,

$$\frac{\eta}{2}\int_{V_d} I^2 dV > \frac{\eta}{2}\int_V I^{o2} dV . \tag{3.15}$$

The following inequality is obtained by adding the respective sides of (3.12) and (3.15).

$$\int_{S_V} T_i v_i dS > \int_{S_V} T_i^* v_i dS - \frac{1}{2\eta}\int_V (J^* - k)^2 dV + \frac{\eta}{2}\int_V I^{o2} dV \tag{3.16}$$

T^* and J^* are obtained from any statically admissible stress field with $J^* \geqslant k$, and I^o is obtained from the velocity solution of the corresponding slow Newtonian flow problem. Inequality (3.16) then gives a lower bound for $\int_{S_V} T_i v_i dS$.

A statically admissible stress field σ_{ij}^* with $J^* > k$ may be obtained from a kinematically admissible strain rate field e_{ij}^+, which is nowhere zero, by solving the non-homogeneous perfectly plastic problem with distribution of shear yield stress

$$k' = k + \eta\sqrt{2e_{ij}^+ e_{ij}^+} \; . \tag{3.17}$$

That is

$$J^* = \sqrt{\frac{1}{2}s^*_{ij}s^*_{ij}} = k + \eta\sqrt{2e_{kl}^+ e_{kl}^+} \; . \tag{3.18}$$

It should be noted that the strain rate field e_{ij}^* associated with σ_{ij}^* according to the constitutive relationships (2.10) is not kinematically admissible and is different from e_{ij}^+; however, their invariants are equal, that is, $\sqrt{2e_{ij}^+ e_{ij}^+} = \sqrt{2e_{ij}^* e_{ij}^*}$. With (3.18) substituted, (3.16) becomes

$$\int_{S_V} T_i v_i dS > \left[\int_{S_V} T_i^* v_i dS - \frac{\eta}{2}\int_V I^{+2} dV\right] + \frac{\eta}{2}\int_V I^{o2} dV. \tag{3.19}$$

One lower bound is easily obtained by letting e_{ij}^+ be equal to the strain rate solution e_{ij}^o of the corresponding slow Newtonian problem. Then the lower bound is given by

$$\int_{S_V} T_i v_i dS > \int_{S_V} T_i^* v_i dS \ , \tag{3.20}$$

where T_i^* is associated with σ_{ij}^* which is obtained from solving the non-homogeneous perfectly plastic problem with shear yield stress distribution equal to $k + \sqrt{2e_{ij}^o e_{ij}^o}$.

Another possibly better lower bound may be obtained by letting e_{ij}^+ be a kinematically admissible strain rate field that is nowhere zero and that depends on some unknown parameters and finding the parameters that maximizes the bracketed expression in (3.19). This maximizing process can be accomplished only if the statically admissible stress field obtained from e_{ij}^+ does not have any stress discontinuities.

An application of the extensions will be presented later in Chapter IV.

CHAPTER IV

VISCO-PLASTIC TORSION

In this chapter, the problem of pure torsion of a right prismatic bar is extended to a homogeneous isotropic visco-plastic material, the Bingham solid. The initial yielding of the solid is assumed to be governed by the Mises yield criterion. Unlike the plate problem solved by Appleby and Prager (10), this problem is not simplified by assuming the Tresca yield criterion or any piece-wise linear yield criterion instead of the Mises yield criterion. For the torsion problem, all isotropic yield criteria that do not exhibit any Bauschinger effect reduce to $\sqrt{s_{xz}^{2} + s_{yz}^{2}} = k$ and the yield locus in the (s_{xz}, s_{yz}) plane is a circle, as shown in Fig. 4.1. Any attempt to linearize this locus results

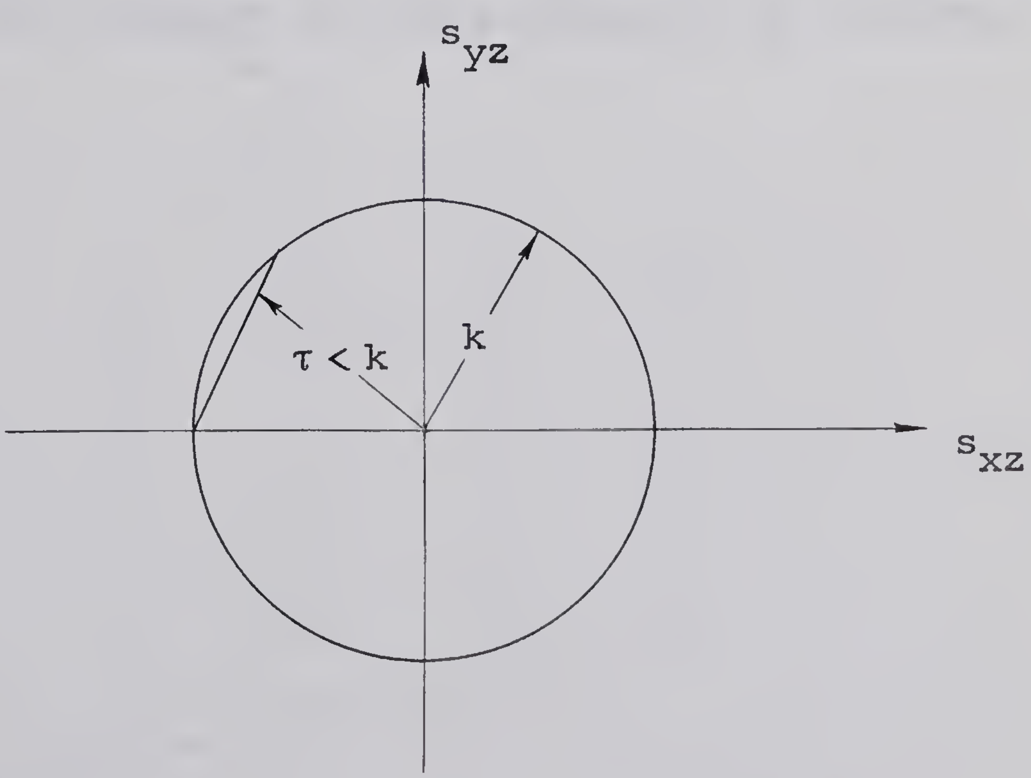

Fig. 4.1 Yield Locus for Pure Torsion

in an anisotropic yield criterion, since a chord drawn on the circle has a distance from the origin smaller than k and, consequently, a yield stress smaller than k is obtained for different directions of the stress vector.

Approximate solutions are obtained for the torsion problem by a semi-inverse procedure, an approximating procedure, and the Ritz's method. The torques evaluated from these approximate solutions are then compared with the upper and lower bound torques obtained from the extensions of the extremum principles presented in Chapter III.

4.1 Statement of Problem

The z-axis is taken parallel to a generator of the bar and one end is fixed in the (x,y) plane as shown in Fig. 4.2. The statement of the problem is as follows:

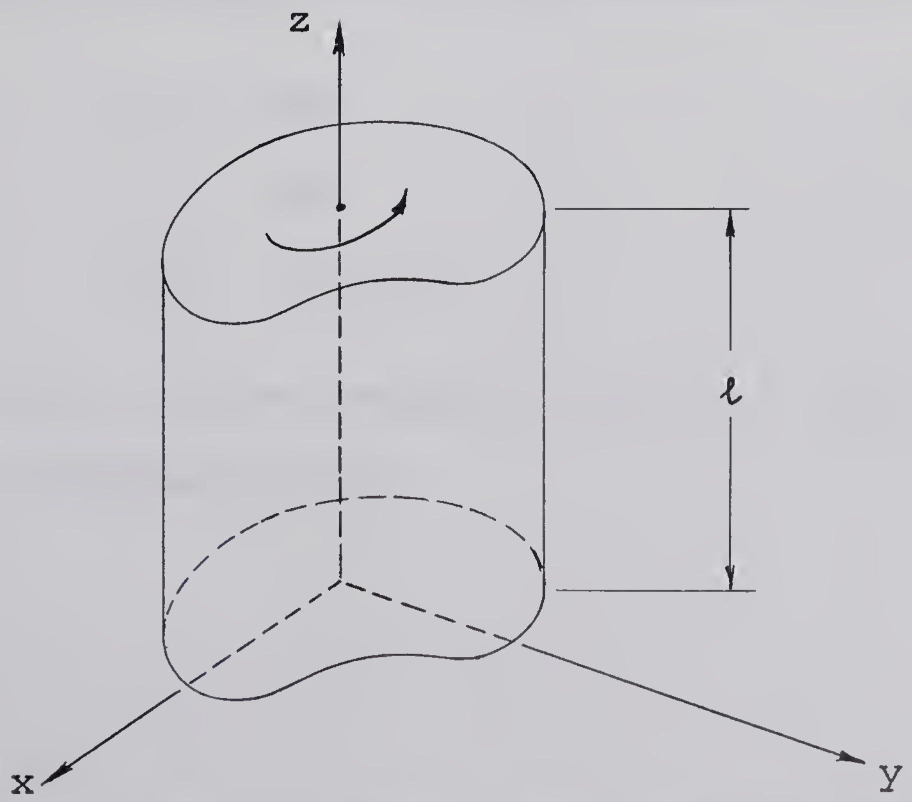

Fig. 4.2 Right Prismatic Bar under Pure Torsion

The bar is twisted by a pure torque in the z-direction and is assumed to be rigid until the torque has exceeded the fully plastic torque. When it starts to deform, the rate of twist per unit length $\dot{\theta}$ is a constant and the ends are free to warp.

4.2 Semi-Inverse Procedure

A semi-inverse procedure, similar to Saint Venant's procedure for the elastic torsion problem, is used to obtain two formulations of the problem, one in terms of the warping function and the other in terms of a stress function. An approximate solution to the problem is obtained by applying a perturbation technique to the formulation in terms of the stress function.

For the semi-inverse procedure in terms of warping function, the following velocity components are assumed:

$$\left.\begin{aligned} u &= -\dot{\theta}zy \\ v &= \dot{\theta}zx \\ w &= w(x,y,\dot{\theta}) \ , \end{aligned}\right\} \tag{4.1}$$

where w is the unknown warping function. The corresponding strain rate components are

$$\left.\begin{aligned} e_{xx} &= e_{yy} = e_{zz} = e_{xy} = 0 \\ e_{xz} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} - \dot{\theta} y\right) \\ e_{yz} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \dot{\theta} x\right) , \end{aligned}\right\} \tag{4.2}$$

and

$$I = 2\sqrt{e_{xz}{}^2 + e_{yz}{}^2} = \left[\left(\frac{\partial w}{\partial x} - \dot{\theta} y\right)^2 + \left(\frac{\partial w}{\partial y} + \dot{\theta} x\right)^2\right]^{\frac{1}{2}} . \tag{4.3}$$

The strain rates e_{xz} and e_{yz} are related by the compatibility equation

$$\frac{\partial e_{xz}}{\partial y} - \frac{\partial e_{yz}}{\partial x} = - \dot{\theta} . \tag{4.4}$$

Substitution of the strain rates (4.2) in the constitutive relationships (2.6) gives

$$s_{xz} = \left[\eta + \frac{k}{\left\{\left(\frac{\partial w}{\partial x} - \dot{\theta} y\right)^2 + \left(\frac{\partial w}{\partial y} + \dot{\theta} x\right)^2\right\}^{\frac{1}{2}}} \right] \left(\frac{\partial w}{\partial x} - \dot{\theta} y\right)$$

(4.5)

$$s_{yz} = \left[\eta + \frac{k}{\left\{\left(\frac{\partial w}{\partial x} - \dot{\theta} y\right)^2 + \left(\frac{\partial w}{\partial y} + \dot{\theta} x\right)^2\right\}^{\frac{1}{2}}} \right] \left(\frac{\partial w}{\partial y} + \dot{\theta} x\right) .$$

These stress components must satisfy the equilibrium equation

$$\frac{\partial s_{xz}}{\partial x} + \frac{\partial s_{yz}}{\partial y} = 0 , \tag{4.6}$$

consequently, the equilibrium equation in terms of the warping function is

$$\eta\nabla^2 w + \frac{k}{I^3}\left[\left(\frac{\partial w}{\partial y} + \dot\theta x\right)^2 \frac{\partial^2 w}{\partial x^2} - 2\left(\frac{\partial w}{\partial y} + \dot\theta x\right)\left(\frac{\partial w}{\partial x} - \dot\theta y\right)\frac{\partial^2 w}{\partial x \partial y}\right.$$
$$\left. + \left(\frac{\partial w}{\partial x} - \dot\theta y\right)^2 \frac{\partial^2 w}{\partial y^2}\right] = 0 . \tag{4.7}$$

This is a quasi-linear elliptic partial differential equation. Since the lateral boundary is stress free, the boundary condition is

$$s_{xz}n_x + s_{yz}n_y = 0 , \tag{4.8}$$

where n_x and n_y are the components of the outward unit normal vector of the boundary curve. From (4.5), this becomes

$$\left(\eta + \frac{k}{I}\right)\left[\left(\frac{\partial w}{\partial x} - \dot\theta y\right)n_x + \left(\frac{\partial w}{\partial y} + \dot\theta x\right)n_y\right] = 0 . \tag{4.9}$$

Since the term in the first bracket is positive definite, the boundary condition to be satisfied by w is

$$\frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y = \dot\theta(yn_x - xn_y) \tag{4.10}$$

or

$$\frac{\partial w}{\partial n} = \dot\theta(yn_x - xn_y) \tag{4.11}$$

at the boundary.

Therefore, the formulated boundary value problem in terms of warping function is equation (4.7) subject to the boundary condition (4.11). With this formulated problem, stress field (4.5) can be shown to be equivalent to a pure torque.

The formulation in terms of the stress function involves the definition of a stress function U(x,y), which is continuous and has piece-wise continuous first derivatives, such that

$$s_{xz} = \frac{\partial U}{\partial y} \tag{4.12}$$

$$s_{yz} = - \frac{\partial U}{\partial x} \quad . \tag{4.13}$$

These stress components automatically satisfies equilibrium equation (4.6). The stress invariant and corresponding strain rates are then

$$J = \sqrt{\frac{1}{2} s_{kl} s_{kl}} = \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 \right]^{\frac{1}{2}} \tag{4.14}$$

$$e_{xz} = \frac{1}{2\eta} \left[1 - \frac{k}{\sqrt{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2}} \right] \frac{\partial U}{\partial y} \tag{4.15}$$

$$e_{yz} = - \frac{1}{2\eta} \left[1 - \frac{k}{\sqrt{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2}} \right] \frac{\partial U}{\partial x} \, . \tag{4.16}$$

These strain rate components must satisfy the compatibility equation (4.4). After substitution of (4.15) and (4.16) in (4.4) and rearranging,

$$\nabla^2 U - \frac{k}{J^3}\left[\left(\frac{\partial U}{\partial y}\right)^2 \frac{\partial^2 U}{\partial x^2} - 2\,\frac{\partial U}{\partial x}\frac{\partial U}{\partial y}\frac{\partial^2 U}{\partial x \partial y} + \left(\frac{\partial U}{\partial x}\right)^2 \frac{\partial^2 U}{\partial y^2}\right] = -\,2\eta\dot{\theta}. \quad (4.17)$$

The boundary condition to be satisfied is obtained from (4.8); thus,

$$\frac{\partial U}{\partial y}\, n_x - \frac{\partial U}{\partial x}\, n_y = 0\ . \quad (4.18)$$

Since $\quad n_x = \dfrac{dy}{ds} \quad (4.19)$

and $\quad n_y = -\dfrac{dx}{ds}\ , \quad (4.20)$

where s is the length measured along the boundary curve, equation (4.18) becomes

$$\frac{\partial U}{\partial y}\frac{\partial y}{ds} + \frac{\partial U}{\partial x}\frac{\partial x}{ds} = \frac{dU}{ds} = 0 \quad (4.21)$$

or $\quad U = \text{constant} \quad$ on the boundary. $\quad (4.22)$

For simply-connected regions, the constant can be taken as zero; therefore,

$$U = 0 \quad \text{on the boundary.} \quad (4.23)$$

Then, the formulated boundary value problem in terms of stress function is equation (4.17) subject to the boundary condition (4.23) or (4.22).

4.3 Visco-Plastic Torsion of a Prismatic Bar of Square Cross-Section

The problem is specialized in this section for a bar with a square square cross-section. Referring to Fig. 4.3, the solutions of the corresponding elastic and plastic problems have symmetry about the x and y axes, and in each quadrant, the warping function and warping velocity are anti-symmetric about the diagonal and the stress trajectories are symmetric about the diagonal (13) (17). It is reasonable to assume that the solution of the visco-plastic problem also possesses these symmetry properties. Therefore, it is possible to consider only one-eighth of the region. For the rest of this chapter, the triangular region AOB is being considered only.

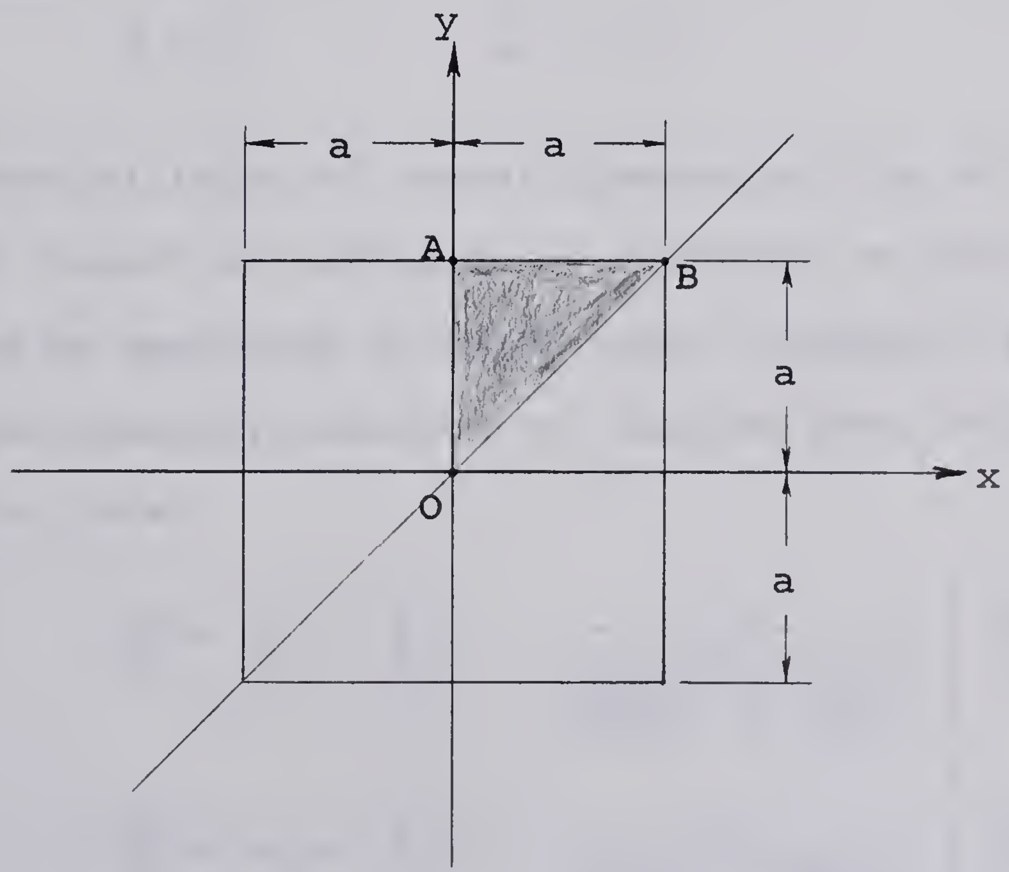

Fig. 4.3 Square Cross-Section of the Bar

The boundary conditions derived in Section 4.2 can now be written for region AOB. For the warping function formulation, the boundary condition (4.11) becomes

$$\frac{\partial w}{\partial y} = - \dot{\theta} x \qquad \text{at} \quad y = a \,. \tag{4.24}$$

To ensure that the warping velocity is continuous throughout the region, w must vanish on OA and OB, because of the foregoing symmetry considerations. Therefore,

$$w = 0 \qquad \text{at } x = 0 \tag{4.25}$$

and

$$w = 0 \qquad \text{at } x = y \,. \tag{4.26}$$

For the stress function formulation, boundary condition (4.23) becomes

$$U = 0 \qquad \text{at } y = a. \tag{4.27}$$

Since the possibilities of stress discontinuities at lines OA and OB cannot be excluded, no condition on the stress function can be specified on these lines. However, the corresponding velocity solution w derived from (4.15) and (4.16) in the form

$$\frac{\partial w}{\partial x} - \dot{\theta} y = \frac{1}{\eta}\left[1 - \frac{k}{\sqrt{(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2}}\right]\frac{\partial U}{\partial y} \tag{4.28}$$

$$\frac{\partial w}{\partial y} + \dot{\theta} x = -\frac{1}{\eta}\left[1 - \frac{k}{\sqrt{(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2}}\right]\frac{\partial U}{\partial x} \tag{4.29}$$

must satisfy (4.25) and (4.26).

4.4 Approximate Solution by Perturbation Method

The viscosity effect in the torsion problem can be represented by a dimensionless parameter $\epsilon = \frac{\eta\dot{\theta}a}{k}$. This parameter gives the ratio of the characteristic viscous shear stress and plastic shear stress. For $\epsilon << 1$, that is when the viscosity effect is small, it is reasonable to assume that the visco-plastic solution is slightly perturbed from the perfectly plastic solution. To find such a solution, a perturbation procedure (18) is used to solve the boundary value problem for stress function, U, stated in the previous section. The following dimensionless variables are introduced:

$$\left.\begin{aligned} \bar{U} &= \frac{U}{ka} \\ \bar{x} &= \frac{x}{a} \\ \bar{y} &= \frac{y}{a} \\ \epsilon &= \frac{\eta\dot{\theta}a}{k} \end{aligned}\right\} \tag{4.30}$$

Equation (4.17) expressed in terms of these dimensionless variables becomes (with the bars dropped*)

$$\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2\right]^{\frac{3}{2}} (\nabla^2 U + 2\epsilon) - \left[\left(\frac{\partial U}{\partial y}\right)^2 \frac{\partial^2 U}{\partial x^2} - 2\frac{\partial U}{\partial x}\frac{\partial U}{\partial y}\frac{\partial^2 U}{\partial x \partial y} + \left(\frac{\partial U}{\partial x}\right)^2 \frac{\partial^2 U}{\partial y^2}\right] = 0. \tag{4.31}$$

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*Refer to nomenclature on p. x.

To proceed with the method, the solution is assumed to be of the series form

$$U = U_o + \epsilon U_1 + \epsilon^2 U_2 + \ldots \quad . \tag{4.32}$$

Upon substituting in (4.31), using the binomial expansion for the power terms and collecting terms of the same order of ϵ, the following equations are obtained:

<u>For zeroth order of ϵ</u>

$$|\nabla U_o|^3 \nabla^2 U_o - \left[\left(\frac{\partial U_o}{\partial y}\right)^2 \frac{\partial^2 U_o}{\partial x^2} - 2 \frac{\partial U_o}{\partial x} \frac{\partial U_o}{\partial y} \frac{\partial^2 U_o}{\partial x \partial y} + \left(\frac{\partial U_o}{\partial x}\right)^2 \frac{\partial^2 U_o}{\partial y^2} \right] = 0 \tag{4.33}$$

<u>For first order of ϵ</u>

$$\begin{aligned} &\left[|\nabla U_o|^3 - \left(\frac{\partial U_o}{\partial y}\right)^2 \right] \frac{\partial^2 U_1}{\partial x^2} + 2 \frac{\partial U_o}{\partial x} \frac{\partial U_o}{\partial y} \frac{\partial^2 U_1}{\partial x \partial y} + \left[|\nabla U_o|^3 - \left(\frac{\partial U_o}{\partial x}\right)^2 \right] \frac{\partial^2 U_1}{\partial y^2} \\ &\quad + \left[3 |\nabla U_o| \nabla^2 U_o \frac{\partial U_o}{\partial x} + 2 \frac{\partial U_o}{\partial y} \frac{\partial^2 U_o}{\partial x \partial y} - 2 \frac{\partial U_o}{\partial x} \frac{\partial^2 U_o}{\partial y^2} \right] \frac{\partial U_1}{\partial x} \\ &\quad + \left[3 |\nabla U_o| \nabla^2 U_o \frac{\partial U_o}{\partial y} + 2 \frac{\partial U_o}{\partial x} \frac{\partial^2 U_o}{\partial x \partial y} - 2 \frac{\partial U_o}{\partial y} \frac{\partial^2 U_o}{\partial x^2} \right] \frac{\partial U_1}{\partial y} \\ &\quad + 2 |\nabla U_o|^3 = 0 \end{aligned} \tag{4.34}$$

<u>And so forth for higher order of ϵ.</u>

In (4.33) and (4.34),

$$\left|\nabla U_o\right| = \left[\left(\frac{\partial U_o}{\partial x}\right)^2 + \left(\frac{\partial U_o}{\partial y}\right)^2 \right]^{\frac{1}{2}} \quad . \tag{4.35}$$

The boundary conditions for the above equations are obtained by substituting series (4.32) into the dimensionless form of (4.27). The resulting boundary conditions are

$$U_o = U_1 = U_2 = . \; . \; . = 0 \qquad \text{at } y = 1. \tag{4.36}$$

The fully plastic stress function solution for the corresponding perfectly plastic problem is $U_p = 1 - y$. This satisfies the zeroth order equation (4.33). Therefore, it is reasonable to take this as the zeroth order solution. Thus,

$$U_o = 1 - y. \tag{4.37}$$

With U_o known, U_1 can then be found from (4.34). Many terms vanish when (4.37) is substituted into (4.34). The final equation for U_1 is

$$\frac{\partial^2 U_1}{\partial y^2} + 2 = 0. \tag{4.38}$$

The solution for this equation is

$$U_1 = - y^2 + y\, f(x) + q(x) \tag{4.39}$$

where $f(x)$ and $q(x)$ are as yet unknown functions of x.

From boundary condition (4.36),

$$q(x) = 1 - f(x). \tag{4.40}$$

Therefore,

$$U_1 = 1 - y^2 + f(x)(y-1). \tag{4.41}$$

The function f(x) must now be determined from the velocity boundary conditions (4.25) and (4.26). In order to do this, the velocity field corresponding to the stress function $U = U_o + \epsilon U_1$ must be obtained from the dimensionless form of (4.28) and (4.29) written below with $\bar{w} = \frac{w}{\dot{\theta} a^2}$. (Bars actually omitted.)

$$\frac{\partial w}{\partial x} - y = \frac{1}{\epsilon}\left[1 - \frac{1}{\sqrt{(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2}}\right]\frac{\partial U}{\partial y} \tag{4.42}$$

$$\frac{\partial w}{\partial y} + x = -\frac{1}{\epsilon}\left[1 - \frac{1}{\sqrt{(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2}}\right]\frac{\partial U}{\partial x} \tag{4.43}$$

$$U = 1 - y + \epsilon\left[1 - y^2 + f(x)(y - 1)\right] \tag{4.44}$$

$$s_{xz} = \frac{\partial U}{\partial y} = -1 + \epsilon\left[-2y + f(x)\right] \tag{4.45}$$

$$s_{yz} = -\frac{\partial U}{\partial x} = -\epsilon f'(x)(y - 1) \tag{4.46}$$

where s_{xz} and s_{yz} are in dimensionless form (i.e., divided by k).

$$\sqrt{(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2} = \left\{1 + 2\epsilon\left[2y - f(x)\right] + \epsilon^2\left[2y-f(x)\right]^2 + \epsilon^2\left[f'(x)(y-1)\right]^2\right\}^{\frac{1}{2}}$$

$$\approx 1 + \epsilon\left[2y - f(x)\right] \tag{4.47}$$

Therefore, upon substitution of (4.44) to (4.47) and using binomial expansion, (4.42) and (4.43) become

$$\frac{\partial w}{\partial x} - y = -2y + f(x) \tag{4.48}$$

$$\frac{\partial w}{\partial y} + x = \epsilon\ f'(x)(y-1)\left[2y - f(x)\right] . \tag{4.49}$$

From (4.48),

$$w = -xy + \int f(x)dx + h(y) , \tag{4.50}$$

where $h(y)$ is an as yet unknown function of y. From boundary condition (4.25), that is, $w = 0$ at $x = 0$,

$$h(y) = \text{constant} = -\left[\int f(x)dx\right]_{x=0} .$$

Therefore,

$$w = -xy + \int_0^x f(z)dz . \tag{4.51}$$

From boundary condition (4.26), that is $w = 0$ at $x = y$,

$$\int_0^x f(z)dz = x^2$$

and

$$f(x) = 2x. \tag{4.52}$$

(4.52) will not satisfy equation (4.49) exactly, but it is satisfied to the order of approximation of the perturbation method, since the left hand side is zero and the right hand side has a maximum error equal to ϵ and a root mean square error of 0.298ϵ. Therefore, for $\epsilon \ll 1$, the approximate solution of the problem may be taken as

$$\left.\begin{aligned} U &= 1 - y + \epsilon\left[1 - y^2 - 2x(1 - y)\right] \\ s_{xz} &= -1 - 2\epsilon\,(y - x) \\ s_{yz} &= 2\epsilon\,(1 - y)\ . \end{aligned}\right\} \tag{4.53}$$

The stress trajectories or constant U lines are shown in Fig. 4.4. The torque in dimensionless form is

$$\overline{T} = \frac{T}{a^3k} = 8\int_{AOB} (s_{yz}x - s_{xz}y)\,dxdy\ . \tag{4.54}$$

With the approximate solution (4.53),

$$\overline{T} = \frac{8}{3}\,(1 + \epsilon). \tag{4.55}$$

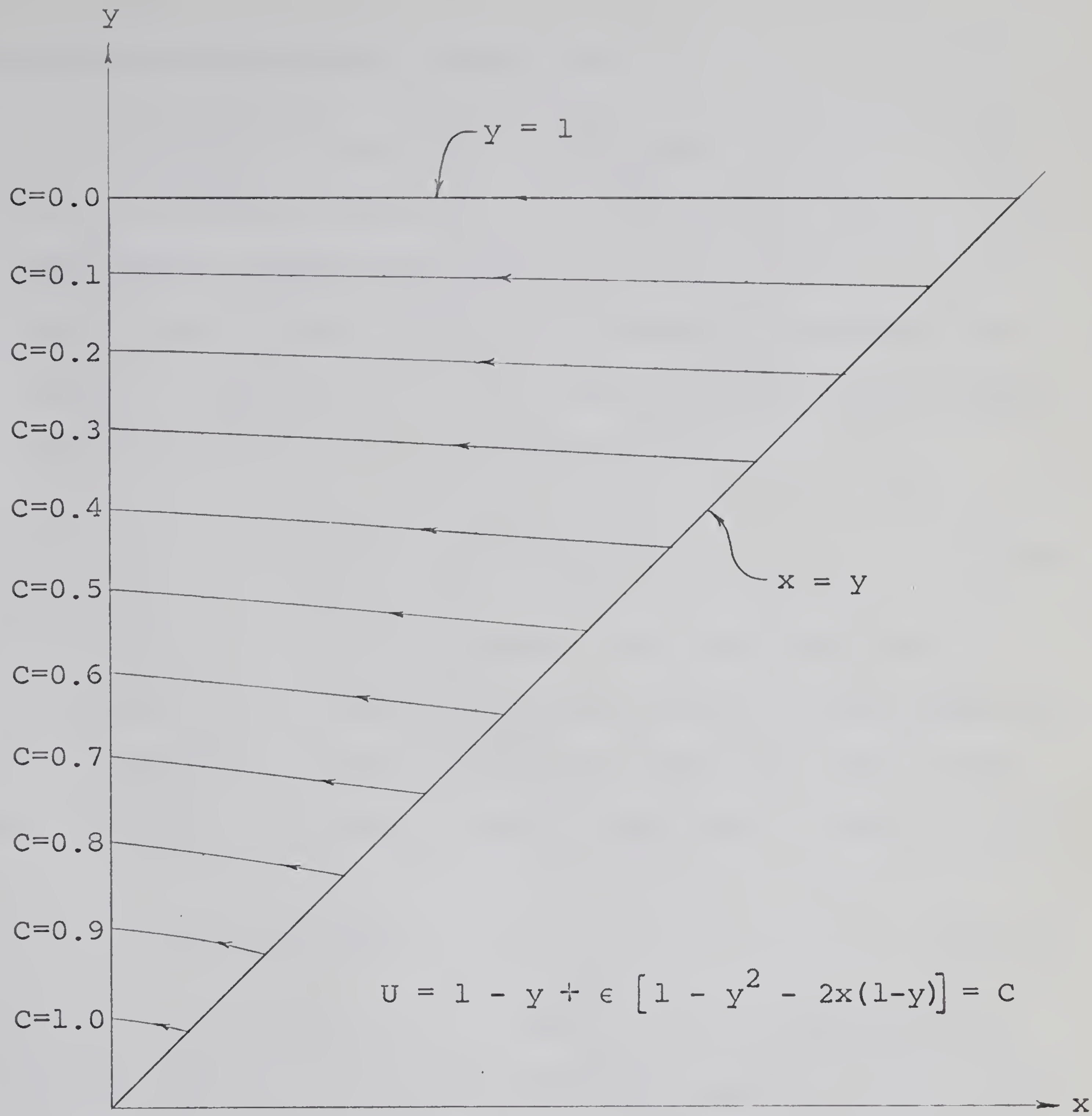

Fig. 4.4 Approximate Stress Trajectories from Perturbation Method

4.5 Approximate Solutions - Method (a)

In order to have other approximate solutions to compare with the solution obtained by the perturbation method, two other approximate methods were applied. The principle of the first approximate method, henceforth referred to as Method (a), is as follows: From (4.5) or (2.6), the magnitude of the shear stress at each point of the cross-section is*

$$\tau = \sqrt{s_{xz}^{\,2} + s_{yz}^{\,2}} = k + \eta I. \tag{4.56}$$

From this equation, it can be deduced that the visco-plastic torsion problem has the same stress solution as the corresponding problem of perfectly plastic torsion of a bar with non-uniform hardness such that the yield stress in shear has a distribution**

$$k' = k + \eta I \ , \tag{4.57}$$

in which I is a function of x and y, and is derived from the velocity solution of the visco-plastic problem. This analogy does not help in getting an exact solution; however, an approximate solution may be obtained by deriving I from the corresponding perfectly plastic velocity solution instead of from the

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*Equations (4.56) to (4.60), inclusive, are in dimensional form. Refer to nomenclature on p. x for the others.

**The velocity field of the non-uniform perfectly plastic problem is the same as that for the visco-plastic problem to within an arbitrary multiplier only.

visco-plastic velocity solution and solving the inhomogeneous perfectly plastic torsion problem, for which a method of solution is known (19), with yield stress distribution

$$k' = k + \eta I_p \ , \tag{4.58}$$

where

$$I_p = \left[\left(\frac{\partial w_p}{\partial x} - \dot{\theta} y\right)^2 + \left(\frac{\partial w_p}{\partial y} + \dot{\theta} x\right)^2 \right]^{\frac{1}{2}} \tag{4.59}$$

and w_p is the velocity solution of the corresponding perfectly plastic problem. For a square section, referring to region AOB in Fig. 4.3,

$$w_p = - \dot{\theta} x(y - x) \ . \tag{4.60}$$

In dimensionless form

$$w_p = -x(y - x) \ . \tag{4.61}$$

Therefore, yield stress distribution (4.58) in dimensionless form is

$$\begin{aligned} \beta = \frac{k'}{k} &= 1 + \epsilon I_p \\ &= 1 + \epsilon \left[\left(\frac{\partial w_p}{\partial x} - y\right)^2 + \left(\frac{\partial w_p}{\partial y} + x\right)^2 \right]^{\frac{1}{2}} \\ &= 1 + 2\epsilon (y - x) \ . \end{aligned} \tag{4.62}$$

The dimensionless shear stress may be written as

$$\begin{aligned} s_{xz} &= - \beta \cos \phi \\ s_{yz} &= \beta \sin \phi \ , \end{aligned} \tag{4.63}$$

where ϕ is the angle between the shear stress vector and

the direction of the negative x-axis. Substituting these stress components into equilibrium equation (4.6) gives

$$\sin\phi \frac{\partial\phi}{\partial x} + \cos\phi \frac{\partial\phi}{\partial y} = \frac{1}{\beta}\left(\cos\phi \frac{\partial\beta}{\partial x} - \sin\phi \frac{\partial\beta}{\partial y}\right) \qquad (4.64)$$

or

$$\frac{\partial\phi}{\partial n} = -\frac{1}{\beta}\left(\cos\phi \frac{\partial\beta}{\partial x} - \sin\phi \frac{\partial\beta}{\partial y}\right), \qquad (4.65)$$

where $\frac{\partial}{\partial n} = -\sin\phi \frac{\partial}{\partial x} - \cos\phi \frac{\partial}{\partial y}$ and n is in the direction normal and 90° counterclockwise from the direction of the stress trajectories, and is in the dimensionless form, that is, divided by a. Since the boundary of the cross-section is a stress trajectory, ϕ is known on it. Therefore, (4.65) may be integrated inward from the boundary and normal to it to determine ϕ for the whole cross-sectional region. The integration of (4.65) or (4.64) is done by a numerical procedure.

Equation (4.64) is used for the numerical procedure, because it is easier to work with square grid points than irregular grid points in which utilization of (4.65) will result. With forward difference for $\frac{\partial\phi}{\partial x}$ and backward difference for $\frac{\partial\phi}{\partial y}$, for a typical grid point (i,j), (4.64) in finite difference form is

$$\tan\phi_{(i,j)} \frac{\phi_{(i+1,j)} - \phi_{(i,j)}}{h} + \frac{\phi_{(i,j)} - \phi_{(i,j-1)}}{h}$$

$$= \frac{1}{\beta_{(i,j)}}\left[\left(\frac{\partial\beta}{\partial x}\right)_{(i,j)} - \tan\phi_{(i,j)}\left(\frac{\partial\beta}{\partial y}\right)_{(i,j)}\right], \qquad (4.66)$$

where h is the grid width. This equation relates the values of ϕ at three grid points $(i+1, j)$, (i,j), and $(i,j-1)$. Referring to Fig. 4.5, the numerical calculation is started at the upper left hand corner of region AOB. Since ϕ is known at grid points A and D, ϕ at grid point C can be determined from (4.62) and (4.66). By proceeding to the right, ϕ for all grid points on line CE can be determined. In the same manner, ϕ for the next grid line below can be determined, and so forth for the whole region.

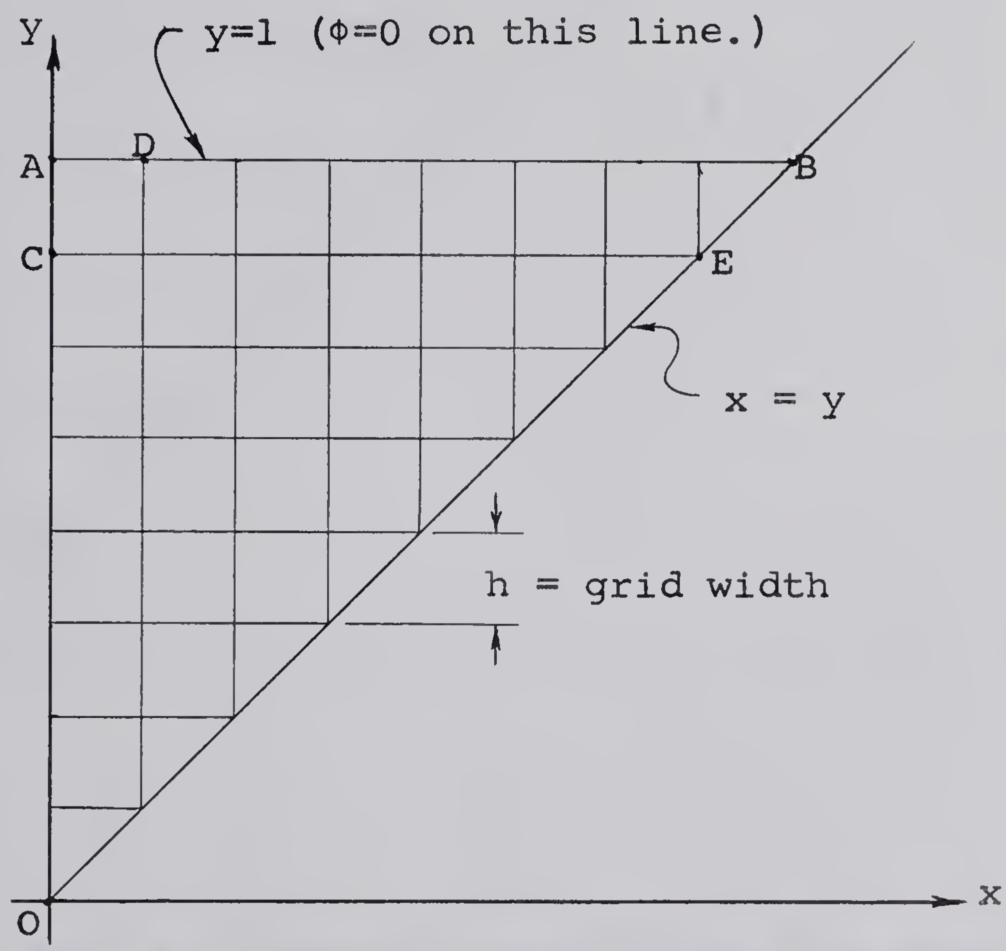

Fig. 4.5 Typical Grided Region for Numerical Calculation

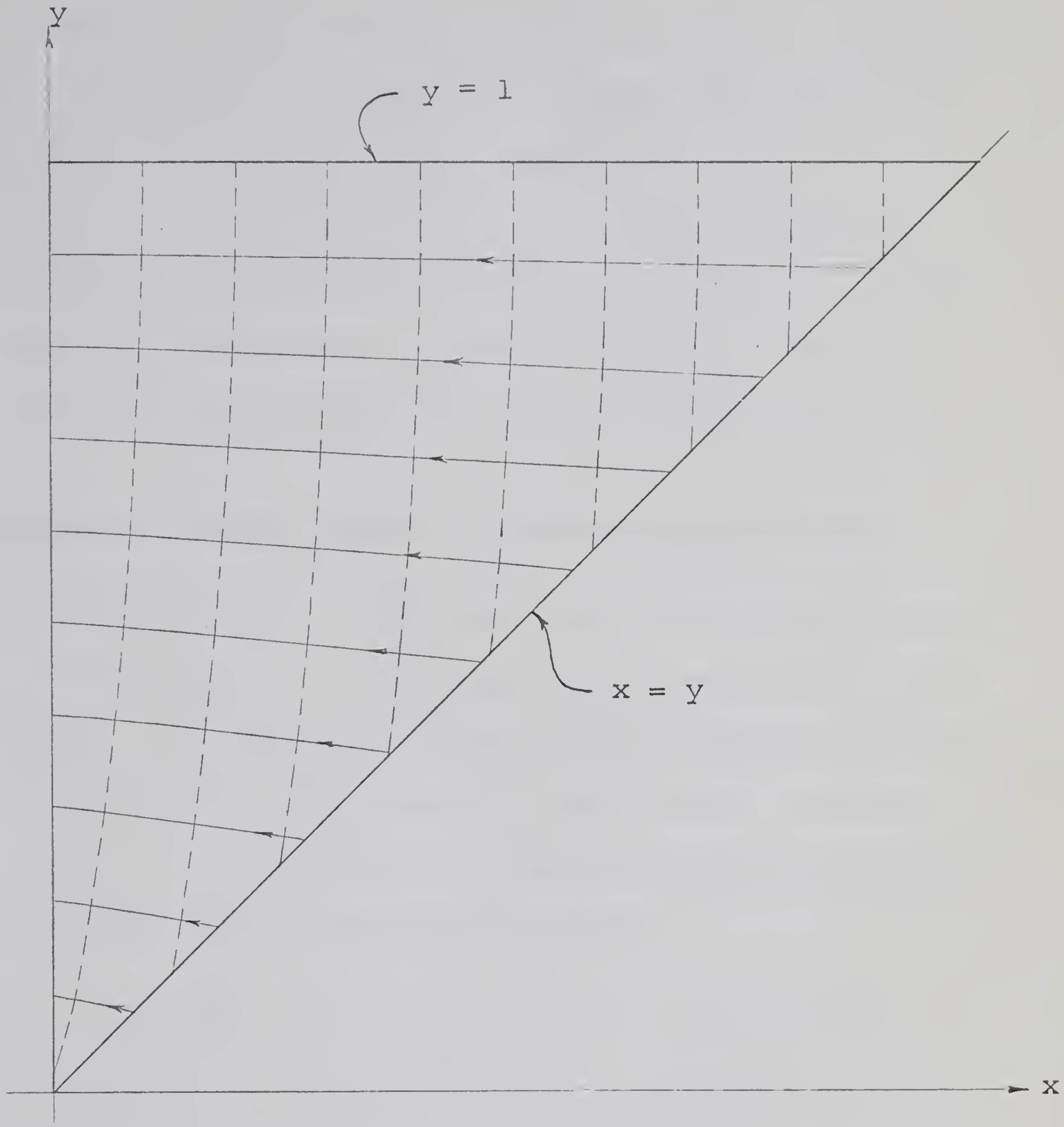

Fig. 4.6 Stress Trajectories from Approximate Method (a)

The above numerical procedure is performed by Computer Program A*, which also calculates the stress components from (4.63) and the dimensionless torque from (4.54). For $\epsilon = 0.1$ and grid size of 0.01 of the width of the square, values of Φ and stresses were calculated and tabulated in Table A.1 of the Appendix, and the dimensionless torque is 2.9281. The approximate stress trajectories are shown in Fig. 4.6.

4.6 Application of Ritz's Method - Approximate Method (b)

Ritz's method is used along with the extremum principle for velocity fields to obtain approximate solutions to the torsion problem. To apply the velocity field extremum principle to the torsion problem, it is assumed that the rate of twist per unit length $\dot{\theta}$ is prescribed by specifying the velocities (referring to Fig. 4.2) in dimensional forms

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\} \quad \text{at } z = 0$$

$$\left.\begin{aligned} u &= -\dot{\theta}\ell y \\ v &= \dot{\theta}\ell x \end{aligned}\right\} \quad \text{at } z = \ell \; .$$

The lateral surface is stress free and, since the ends are free to warp, the normal component of stress on the ends is prescribed

* All the computer programs are given in the Appendix.

to be zero. Also the body forces due to the weight of the material are assumed to be zero. From (2.24) the functional to be minimized is (in dimensional form)

$$H^* = \int_V (\eta I^{*2} + 2kI^*)\,dV \,, \tag{4.67}$$

where I^* is obtained from a kinematically admissible strain rate field. For this pure torsion problem, $dV = \ell dS$, where ℓ is the length of the bar. Since it is simpler to work with dimensionless quantities, the following dimensionless variables are introduced:

$$\bar{H}^* = \frac{H^*}{\dot{\theta} a^4 k} \,, \qquad \bar{I} = \frac{I}{\dot{\theta} a} \,, \qquad d\bar{V} = \frac{dV}{a^3} \,,$$
$$\bar{\ell} = \frac{\ell}{a} \,, \qquad d\bar{s} = \frac{ds}{a^2} \tag{4.68}$$

Therefore, the dimensionless form of (4.67) is (with the bars dropped)

$$H^* = \ell \int_S (\epsilon I^{*2} + 2I^*)\,dS \tag{4.69}$$

where

$$I^* = \left[\left(\frac{\partial w^*}{\partial x} - y\right)^2 + \left(\frac{\partial w^*}{\partial y} + x\right)^2 \right]^{\frac{1}{2}} \tag{4.70}$$

Two kinematically admissible velocity fields are chosen. They are

$$w^* = \gamma w_p \tag{4.71}$$
$$w^* = \gamma w_p + (1 - \gamma) w_v \tag{4.72}$$

where w_p is the fully plastic velocity solution for the corresponding perfectly plastic problem; w_v is the velocity solution for the corresponding Newtonian flow problem; and γ is an unknown multiplication factor to be chosen so that H^* is minimized. Functions depending on only one parameter are chosen, because the highly non-linear nature of the functional (4.69) would make the minimizing process quite difficult for more than one parameter. Henceforth, the application of Ritz's method using velocity field (4.71) will be referred to as Method (b-1) and that using velocity field (4.72) as Method (b-2).

To apply Method (b-1), velocity field w_p for region AOB from (4.61) is substituted in (4.71) giving

$$w^* = -\gamma x(y - x) \ . \tag{4.73}$$

The functional (4.67) for this velocity field and for a square section is

$$G = \frac{H^*}{8\ell} = \epsilon \int_{AOB} \left\{ \left[(1+\gamma) y - 2\gamma x \right]^2 + (1-\gamma)^2 x^2 \right\} dS$$
$$+ 2 \int_{AOB} \left\{ \left[(1+\gamma) y - 2\gamma x \right]^2 + (1-\gamma)^2 x^2 \right\}^{\frac{1}{2}} dS. \tag{4.74}$$

It is easier to find the value of γ that minimizes G by numerically integrating (4.74). For $\epsilon = 0.1$, the minimizing process was done by a computer program and found that G is minimum at $\gamma = 1.0$.

The dimensionless stresses associated with the velocity field (4.73) with $\gamma = 1.0$ are

$$\left.\begin{aligned} s_{xz} &= -1 - 2\epsilon(y - x) \\ s_{yz} &= 0 \end{aligned}\right\} \qquad (4.75)$$

and are equivalent to a pure torque. With this stress field, the dimensionless torque obtained from (4.54) is

$$T = \frac{8}{3}\left(1 + \frac{\epsilon}{2}\right) . \qquad (4.76)$$

To apply Method (b-2), the pure viscous velocity solution for slow Newtonian flow in the torsion problem is obtained from replacing twist by rate of twist in the elastic warping function. Hence, in dimensionless form

$$w_v = xy - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\sinh k_n y}{\cosh k_n} \sin k_n x , \qquad (4.77)$$

where $k_n = \frac{\pi}{2}(2n+1)$. Then (4.72) becomes

$$w^* = -\gamma x(y-x) + (1-\gamma)\left[xy - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\sinh k_n y}{\cosh k_n} \sin k_n x\right] . \qquad (4.78)$$

The velocity field (4.78) was substituted in (4.69) and (4.70), and the integral evaluated numerically by a computer program for

values $0.90 \leq \gamma \leq 1.00$ and $\epsilon = 0.1$. Since the series in (4.78) converges quite rapidly, it was approximated only by the first eight terms in the evaluation. The results showed that H^* is minimum at $\gamma = 0.96$ within the accuracy of the numerical integration. With $\gamma = 0.96$ substituted into (4.78),

$$\frac{\partial w}{\partial x} - y = -0.96(y - x) - \frac{0.64}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh k_n y}{\cosh k_n} \cos k_n x \quad (4.79)$$

$$\frac{\partial w}{\partial y} + x = 0.08x - \frac{0.64}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\cosh k_n y}{\cosh k_n} \sin k_n x \,. \quad (4.80)$$

The corresponding dimensionless stresses are then

$$s_{xz} = (\epsilon + \frac{1}{I})(\frac{\partial w}{\partial x} - y) \quad (4.81)$$

$$s_{yz} = (\epsilon + \frac{1}{I})(\frac{\partial w}{\partial y} + x) \quad (4.82)$$

The dimensionless torque can then be evaluated by (4.54). The numerical values of the stresses and the torque may be obtained numerically by Computer Program B. For $\epsilon = 0.1$ and grid size of 0.01 of the width of the square, values of stresses were evaluated and tabulated in Table A.2, and the dimensionless torque is equal to 2.9293. Approximate stress trajectories are drawn from the values of the stresses and shown in Fig. 4.7.

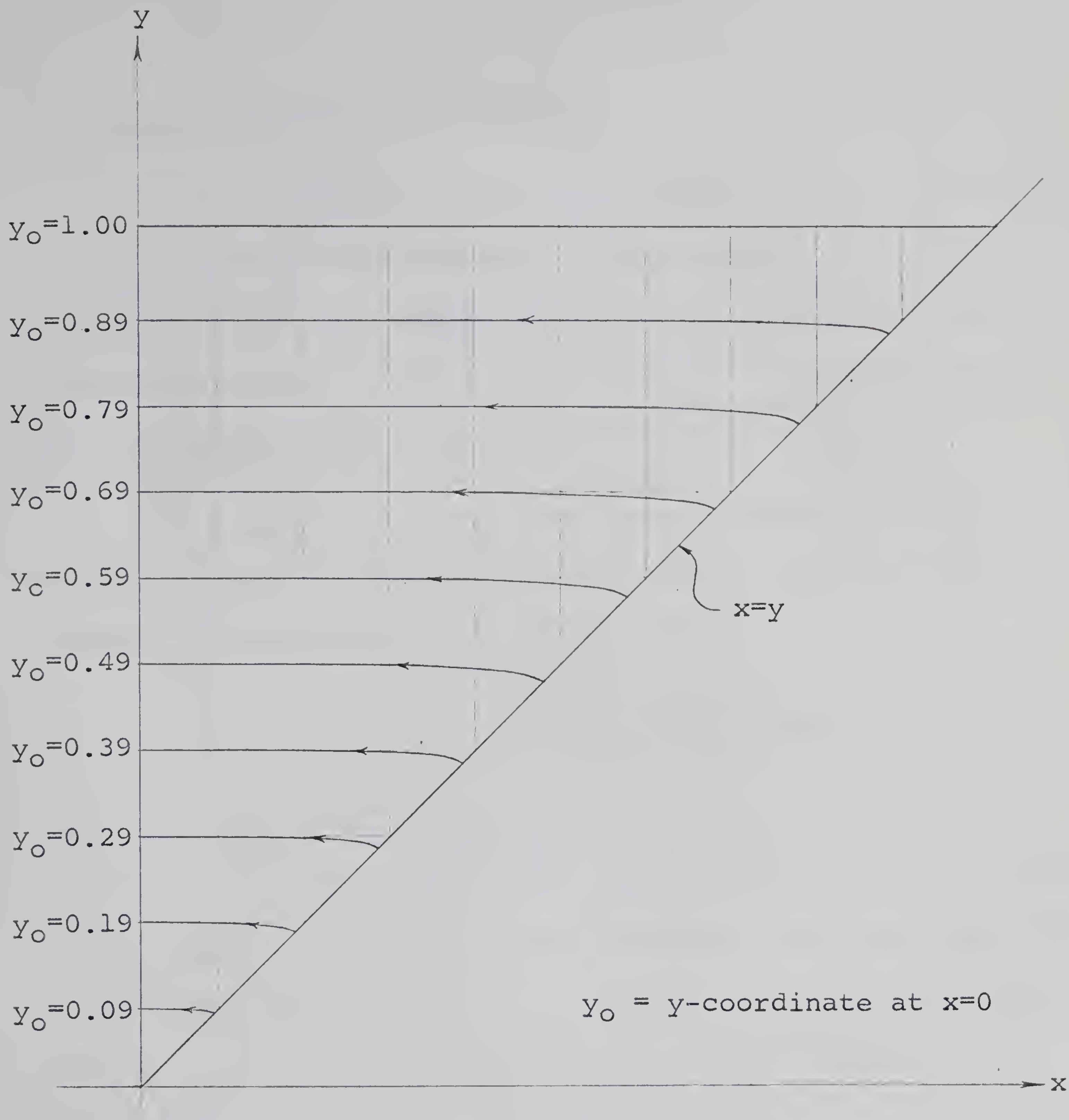

Fig. 4.7 Stress Trajectories from Approximate Method (b-2)

4.7 Upper and Lower Bound Torques

The extensions of the extremum principles presented in Chapter III are applied in this section to the visco-plastic torsion of a bar with square cross-section to obtain upper and lower bound torques. The results are used for comparison with the other approximate torques previously obtained.

To apply the extension of the minimum principle for velocity fields inequality (3.6) is written for the torsion problem in the dimensionless form as follows:

$$\int (\epsilon I^{*2} + I^{*})\,dS > \int (s_{yz}x - s_{xz}y)\,dS, \tag{4.83}$$

where the integrations are over the cross-sectional area of the bar. The right hand side is the dimensionless torque for the torsion problem shown in (4.54); therefore, the left hand side is the upper bound torque. I^{*} must be evaluated from the perfectly plastic velocity solution, w_p. From (4.61), $w_p = -x(y-x)$ in region AOB; consequently,

$$I^{*} = \left[\left(\frac{\partial w_p}{\partial x} - y\right)^2 + \left(\frac{\partial w_p}{\partial y} + x\right)^2 \right]^{\frac{1}{2}} = 2(y-x) \;. \tag{4.84}$$

The upper bound torque is then

$$T_u = 8 \int_{AOB} (\epsilon I^{*2} + I^{*})\,dS$$

$$= 8 \int_0^1 \int_x^1 \left[4\epsilon(y-x)^2 + 2(y-x)\right] dydx = \frac{8}{3}\,(1+\epsilon) \;. \tag{4.85}$$

This is a close upper bound torque for $\epsilon << 1$. For larger values of ϵ, that is if the viscous effects are as much as or much greater than the plastic yield effects, a better upper bound torque is obtained from inequality (3.8) by evaluating I^* from the velocity solution of the corresponding Newtonian flow problem and I^+ from that of the perfectly plastic problem. For the torsion problem, inequality (3.8) in the dimensionless form is

$$\int(\epsilon I^{*2} + 2I^*)\,dS - \int I^+ dS > \int(s_{yz}x - s_{xz}y)\,dS. \qquad (4.86)$$

in which the left-hand side is the upper bound torque. From (4.84), $I^+ = 2(y-x)$, and I^* is evaluated from the velocity field w_v in (4.77). With the first integral numerically integrated, the upper bound torque for large value of ϵ is

$$T_u = 2.249\,\epsilon + 2.918. \qquad (4.87)$$

The torque evaluated from the corresponding perfectly plastic solution is a trivial lower bound for the visco-plastic problem. For the square section, the dimensionless trivial lower bound torque is $8\int_{AOB} ydydx = \frac{8}{3}$.

A better lower bound torque may be obtained from the extension of the maximum principle for stress. The inequality (3.20) written for the torsion problem is (in dimensionless form)

$$\int(s_{yz}x - s_{xz}y)\,dS > \int(s_{yz}{}^*x - s_{xz}{}^*y)\,dS, \qquad (4.88)$$

where the integrations are over the cross-sectional area of the bar. The left hand side is again the dimensionless torque for the torsion problem. Thus, the right hand side is a lower bound. s_{xz}^* and s_{yz}^* are obtained by solving the inhomogeneous perfectly plastic problem with shear yield stress distribution (in dimensionless form)

$$\beta^o = 1 + \epsilon I^o, \tag{4.89}$$

$$I^o = \left[\left(\frac{\partial w_v}{\partial x} - y\right)^2 + \left(\frac{\partial w_v}{\partial y} + x\right)^2 \right]^{\frac{1}{2}}, \tag{4.90}$$

where w_v is the velocity solution of the corresponding slow Newtonian flow problem given in (4.77). The method of solution for the inhomogeneous problem was presented in Section 4.5. Following the same numerical procedure, Computer Program C calculates the stress components s_{xz}^* and s_{yz}^* and evaluates the integral on the right hand side of (4.88). For $\epsilon = 0.10$ and grid size of 0.01 of width of the square, the dimensionless lower bound torque is 2.8950.

To give some indications of how close the upper and lower bound torques enclose the actual torque, upper and lower bound torques were calculated from (4.85) and Computer Program C for the range $0.05 \leqslant \epsilon \leqslant 0.60$, and from (4.87) and Computer Program C for the range $0.6 \leqslant \epsilon \leqslant 12.0$. The results are tabulated in Tables 4.1 and 4.2.

Table 4.1

Upper and Lower Bound Torques for $\epsilon << 1$

ϵ	Upper Bound Torque	Lower Bound Torque	$\frac{\text{Difference}}{\text{Average of Bounds}}$
0.05	2.8000	2.7809	0.68%
0.10	2.9333	2.8950	1.31%
0.15	3.0667	3.0091	1.91%
0.20	3.2000	3.1231	2.43%
0.25	3.3333	3.2370	2.93%
0.30	3.4667	3.3509	3.40%
0.35	3.6000	3.4647	3.83%
0.40	3.7333	3.5784	4.24%
0.45	3.8667	3.6921	4.62%
0.50	4.0000	3.8058	4.98%
0.55	4.1333	3.9193	5.32%
0.60	4.2667	4.0329	5.63%

Table 4.2

Upper and Lower Bound Torques for Large ϵ

ϵ	Upper Bound Torque	Lower Bound Torque	Difference / Average of Bounds
0.50	4.0425	3.8058	6.03%
1.00	5.1670	4.9401	4.49%
1.50	6.2915	6.0717	3.56%
2.00	7.4160	7.2015	2.93%
2.50	8.5405	8.3303	2.49%
3.00	9.6650	9.4581	2.16%
3.50	10.7895	10.5853	1.91%
4.00	11.9140	11.7120	1.71%
4.50	13.0385	12.8383	1.55%
5.00	14.1630	13.9642	1.41%
5.50	15.2875	15.0899	1.30%
6.00	16.4120	16.2153	1.21%
6.50	17.5365	17.3406	1.12%
7.00	18.6610	18.4657	1.05%
7.50	19.7855	19.5906	0.99%
8.00	20.9100	20.7154	0.94%
8.50	22.0345	21.8401	0.89%
9.00	23.1590	22.9647	0.84%
9.50	24.2835	24.0892	0.80%
10.00	25.4080	25.2136	0.77%
10.50	26.5325	26.3380	0.74%
11.00	27.6570	27.4623	0.71%
11.50	28.7815	28.5865	0.68%
12.00	29.9060	29.7107	0.66%

4.8 Summary of Results and Discussion

The approximate solutions are presented in terms of the stress fields (s_{xz} and s_{yz}), the approximate stress trajectories, and the torques evaluated from the stress fields. They are given in terms of dimensionless variables so that the solutions, especially the numerical solutions, can be applicable to different characteristic dimensions, yield stresses, and viscosity coefficients. The stress components are either in the analytical form presented under each section or in the numerical form presented in the Appendix. The stress trajectories are shown in figures under each section. The various torques evaluated are summarized in Table 4.3. To recapitulate, Method (a) is by solving the corresponding inhomogeneous perfectly plastic problem, Method (b-1) is Ritz's method using velocity fields $w^* = \gamma w_p$, and Method (b-2) is Ritz's method using velocity fields $w^* = \gamma w_p + (1 - \gamma)\, w_v$.

Table 4.3

Summary of Torques

(For $\epsilon = 0.1$)

Method of Solution	Dimensionless Torque T/a^3k
Upper Bound	2.9333
Perturbation Method	2.9333
Method (b-2)	2.9293
Method (a)	2.9281
Lower Bound	2.8950
Method (b-1)	2.8000
Trivial Lower Bound	2.6667

One desirable condition for the approximate solution to be a good approximation is that the torque evaluated from it lies within the upper and lower bound torques obtained from the extremum principles. The solution obtained by Method (b-1) gives a torque that is lower than the lower bound torque, and its stress field has $s_{yz} = 0$, which would not be expected in the exact solution. Therefore, the solution obtained from Method (b-1) is probably not a good approximate solution.

The solution obtained by the perturbation method gives a torque that is equal to the upper bound torque; also it has s_{yz} independent of x, which in the exact solution would be dependent on x. Therefore, the perturbation solution is probably not as good an approximate solution as those obtained by Method (a) and Method (b-2).

In the other two solutions, s_{yz} at points near the y-axis are smaller than those at points near the diagonal. This is reasonable, because the y-components of the strain rate due to the twisting are larger near the diagonal than those near the y-axis.

In the solution obtained by Method (a), at points near the diagonal, s_{yz} increases monotonically from the corner to the origin. In the exact solution, s_{yz} near the diagonal would probably increase from zero at the origin, reach a maximum near the mid-point between the origin and the corner, and decrease

towards the corner. This is so, because the rate effects due to the twisting would be greater away from the origin; but hear the corner, because of the influence of the geometry of the section, s_{yz} would become smaller again. The distribution of s_{yz} near the diagonal in the solution obtained by Method (b-2) is similar to that one would expect for the exact solution. Therefore, for the torsion problem, Method (b-2) is probably the best among the four approximate methods considered. However, this does not mean that this method would give good approximate solution for other two dimensional visco-plastic problems.

In conclusion, an approximate solution may be obtained for the torsion problem by the Ritz's method using a kinematically admissible velocity field formed by the linear combination of the velocity solutions of the corresponding perfectly plastic and slow Newtonian problems, or by the approximate method (a) suggested in this chapter. A better approximate solution may probably be obtained by the combination of these two methods.

CHAPTER V

EXPERIMENTAL INVESTIGATIONS OF VISCO-PLASTIC TORSION

The experimental investigation of the pure torsion of circular cylinders of pure lead, which is known to exhibit rate effects at room temperature, is presented in this chapter. The dependence of the yield stress of pure lead on strain rates for various temperatures and strains has been investigated by Loizou and Sims (20) using the compression test. Some effective stress-effective strain curves obtained by compressing commercially pure lead at various strain rates were also included in the investigation of Kobayashi, et al. (21). Davis and Hunter (22) and Johnson (23) have attempted to assess the strain rate sensitivity of materials by using dynamic indentation tests, but lead was not considered. All the investigations mentioned above did not consider a visco-plastic model and the strain rate sensitivity data presented are characteristic of the experiments performed; consequently, the application of the data to other problems is inconvenient and possibly difficult. In the compression test, the definition of stress and strain rate is difficult, and in both methods of testing frictional effects are difficult to eliminate.

The purpose of this experimental work is to investigate the characteristics of pure lead circular cylindrical specimens under pure torsion, and to consider the possibility

of assessing the strain rate sensibility of commercially pure lead by representing the lead by the Bingham model and finding the viscosity coefficient. The advantages of using the torsion test are the absence of frictional problems, the ease of definition of stress and strain, and the ease of solution of the theoretical problem for a Bingham solid. The disadvantage is that the stresses and strains are non-uniform.

This experimental work also investigated briefly the shortening of a pure lead specimen during torsion. Swift (24) and Hill (25) pointed out that a lead cylinder shortens during torsion, and Swift presented some data on the lengthenings of circular cylinders of materials like steel, copper, and aluminum, but tested only one lead specimen.

5.1 Theoretical Consideration.

The solution of the problem of pure torsion of a circular cylindrical bar of an elastic-Bingham solid* under constant rate of twisting is easily obtained. The shear stress distribution is as follows:

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* That is, a solid that behaves as a Hookean solid when the stress state is under the yield stress and as a Bingham solid when the stress state exceeds the yield stress.

$$\tau = \mu\theta r \qquad \text{if } 0 \leqslant r \leqslant R_b \tag{5.1}$$

$$\tau = \eta\dot{\theta} r + k \quad \text{if } R_b < r \leqslant R \tag{5.2}$$

where τ is the magnitude of the shear stress vector, R is the radius of the bar, R_b is the radius of the elastic-plastic boundary, and the shear stress vector is directed tangential to the circle of radius r. R_b is the radius at which τ is equal to k; therefore, from (5.1)

$$R_b = \frac{k}{\mu\theta} \tag{5.3}$$

It can be noted from (5.1) and (5.2) that there is a stress discontinuity at R_b. The torque evaluated from (5.1) and (5.2) is

$$T = \int \tau \, r ds = 2\pi \int_0^R \tau r^2 dr$$

$$= \frac{1}{2}\pi\dot{\theta}\eta(R^4 - R_b{}^4) + \frac{2}{3}\pi k(R^3 - R_b{}^3) + \frac{1}{2}\pi\theta\mu R_b{}^4 \tag{5.4}$$

From (5.3), it follows that

$$T = \frac{1}{2}\pi\dot{\theta}\eta\left[R^4 - \left(\frac{k}{\mu\theta}\right)^4\right] + \frac{2}{3}\pi k\left[R^3 - \frac{1}{4}\left(\frac{k}{\mu\theta}\right)^3\right] . \tag{5.5}$$

For pure lead, $\frac{k}{\mu}$ is approximately 1/390; therefore, for large angle of twist such that $\frac{k}{\mu\theta} << R$, (5.5) becomes

$$T = \frac{1}{2}\pi\dot{\theta}\eta R^4 + \frac{2}{3}\pi k R^3 \quad . \tag{5.6}$$

From this relation, η may be determined by twisting specimens of the same diameter at various constant rates of twist and measuring the torque after a certain angle of twist such that $\frac{k}{\mu\theta} \ll R$. If pure lead behaves as a Bingham solid, the data should supply a linear relation

$$T = A\dot{\theta} + B \quad .$$

Therefore, from the constant A, the coefficient of viscosity η may be determined.

5.2 Experimental Apparatus

The experiments consisted essentially of twisting circular lead bars at various rates of twist and measuring the torques during twist. A variable speed torsion testing machine was designed to perform the experiments. The machine consisted of five main parts:

(1) The Frame and Twisting Heads - The main frame of the machine consisted of the base frame supporting a base plate and four posts mounted vertically on the base plate supporting the top plate. The twisting heads were five-inch three-jaw chucks and were arranged and mounted as shown in Fig. 5.1. The shaft of the moving head was in two parts splitting at the point indicated in Fig. 5.1. This permitted the specimen to shorten or lengthen while being twisted. A photograph of the frame and twisting heads is shown in Fig. 5.3.

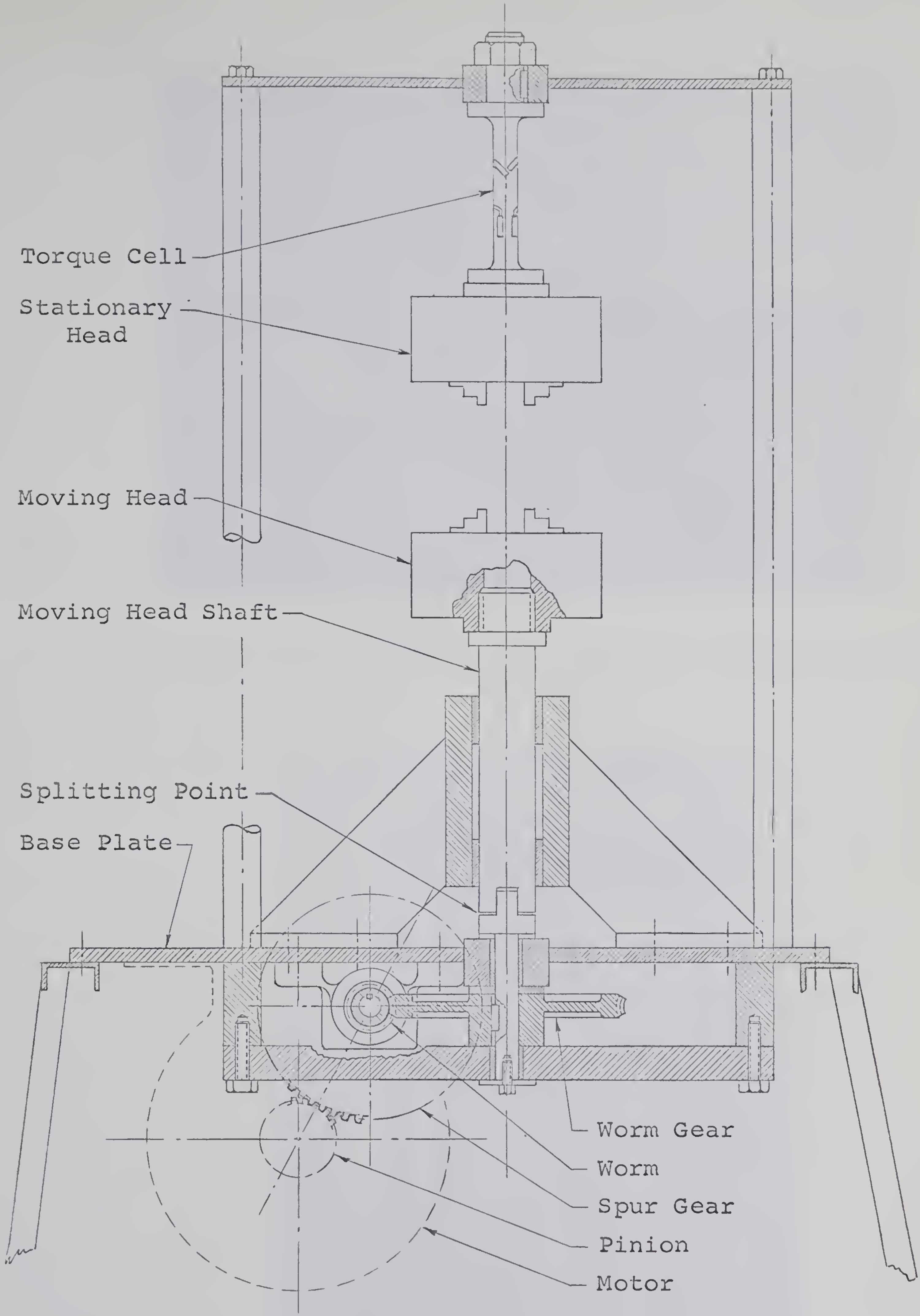


Fig. 5.1 Section View of the Main Part of the Variable Speed Torsion Testing Machine

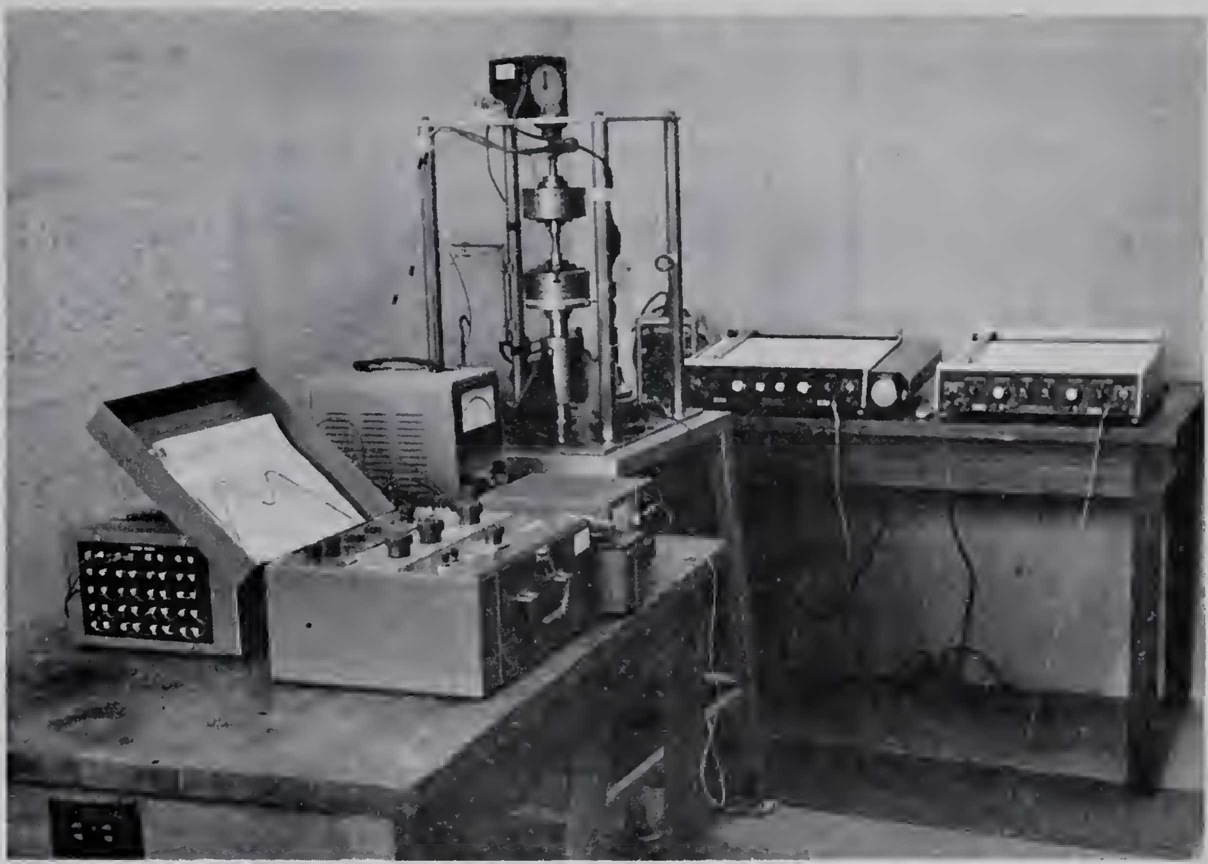


Fig. 5.2 Overall View of the Torsion Testing Machine

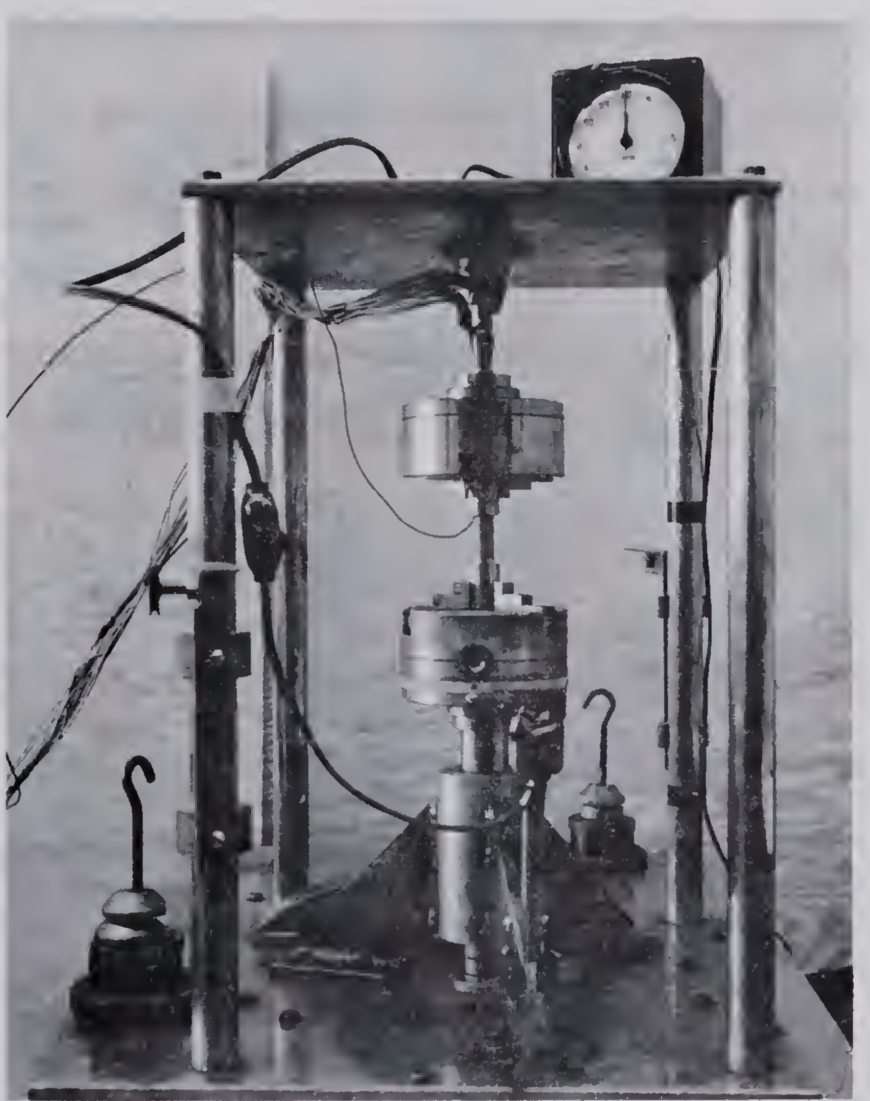


Fig. 5.3 Frame and Twisting Heads

(2) Drive and Speed Adjustment - The drive consisted of two sets of gears: a set of worm (one tooth) and worm gear (60 or 30 teeth), and a set of spur gear (60 teeth) and pinion (20 teeth). The gears and the motor, arranged as shown in Fig. 5.1, gave gear reductions of 1:180 or 1:90 depending on which worm gear was used.

A 3/4 horsepower d.c. motor was used. A much larger motor than required was chosen in order to have a constant speed during testing without the necessity of a feedback control system. Without feedback control, the speed was kept constant to within ± 1.5% by two methods of speed adjustment. One method was for high speed operation, at 1200 rpm to 1800 rpm; a rheostat was connected in series with the armature and the speed was adjusted by varying the rheostat resistance. The other method was for low speed operation, at 180 rpm to 1400 rpm; the armature was connected to an independently adjustable voltage supplied by a "Variac" variable transformer through a rectifier, and the speed was adjusted by varying the "Variac" voltage. The wiring, as shown in Fig. 5.6, was such that changing from one method of speed adjustment to the other only involved throwing a switch. The speed adjustment equipment are shown in the center background of Fig. 5.4.

(3) Speed Measurement - The speed of the moving head was monitored for constancy by a d.c. generator type tachometer connected to the motor shaft. The tachometer output was fed into the y-channel of an x-y recorder, which had the x-channel in time sweep. The actual speed of the moving head was measured by recording the time lapse for each revolution. A projection on the moving head tripped a microswitch (mounted on one leg of the tripod support as shown in Fig. 5.3) connected to an alternate on-off type relay, which starts and stops an electric interval timer (clock). The relay and timer are shown on top of the machine in Fig. 5.4. On the first time around, the clock was started; after one revolution the projection tripped the microswitch again and the clock was stopped. The speed measured by this method was within $\pm$ 1.5%.

(4) Torque Measurement - The torque was measured with a "torque cell" connected in series with the stationary head. The torque cell was a steel spindle with two sets of strain gages on it. One set of four on the spindle at 45 degrees to the axis measured the shear strain during torsion. The other set of four on the spindle parallel to the axis measured the possible strain due to bending. The latter set was installed for preliminary work only; it was used to check for bending during the test and to check the effect of the bending on the torque readings. It was found that the bending was negligible and did not influence the torque measurements. The four strain

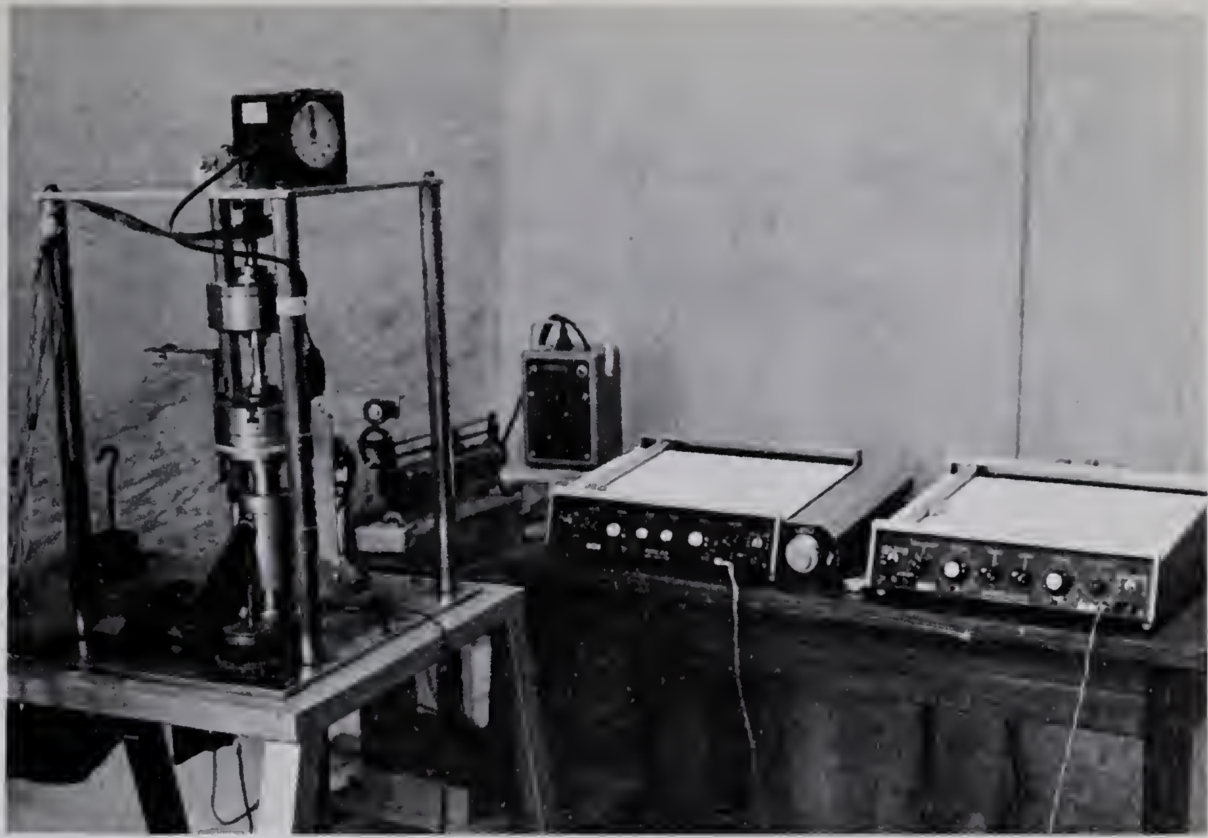


Fig. 5.4 Another View of the Testing Machine

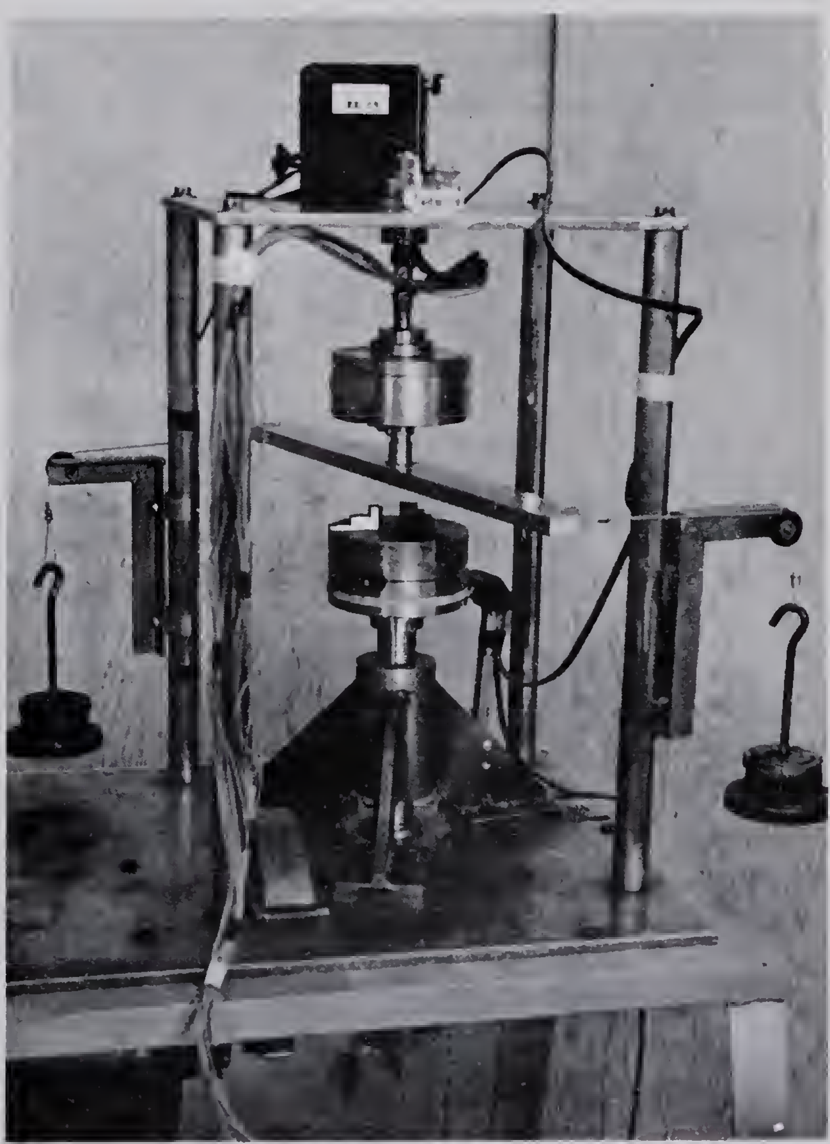


Fig. 5.5 Arrangement for Calibration of the Torque Cell

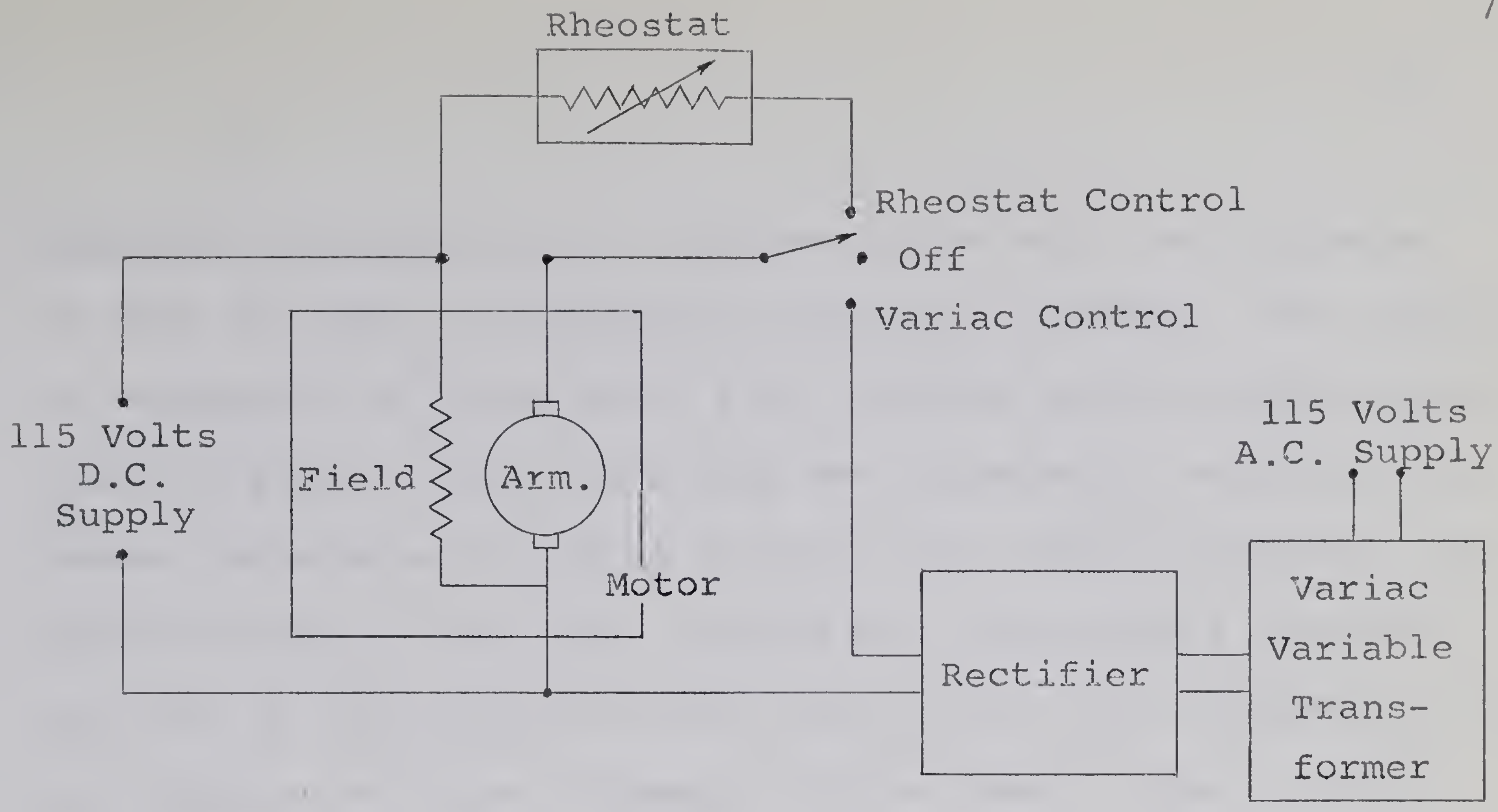


Fig. 5.6 Circuit Diagram for Motor Speed Adjustments

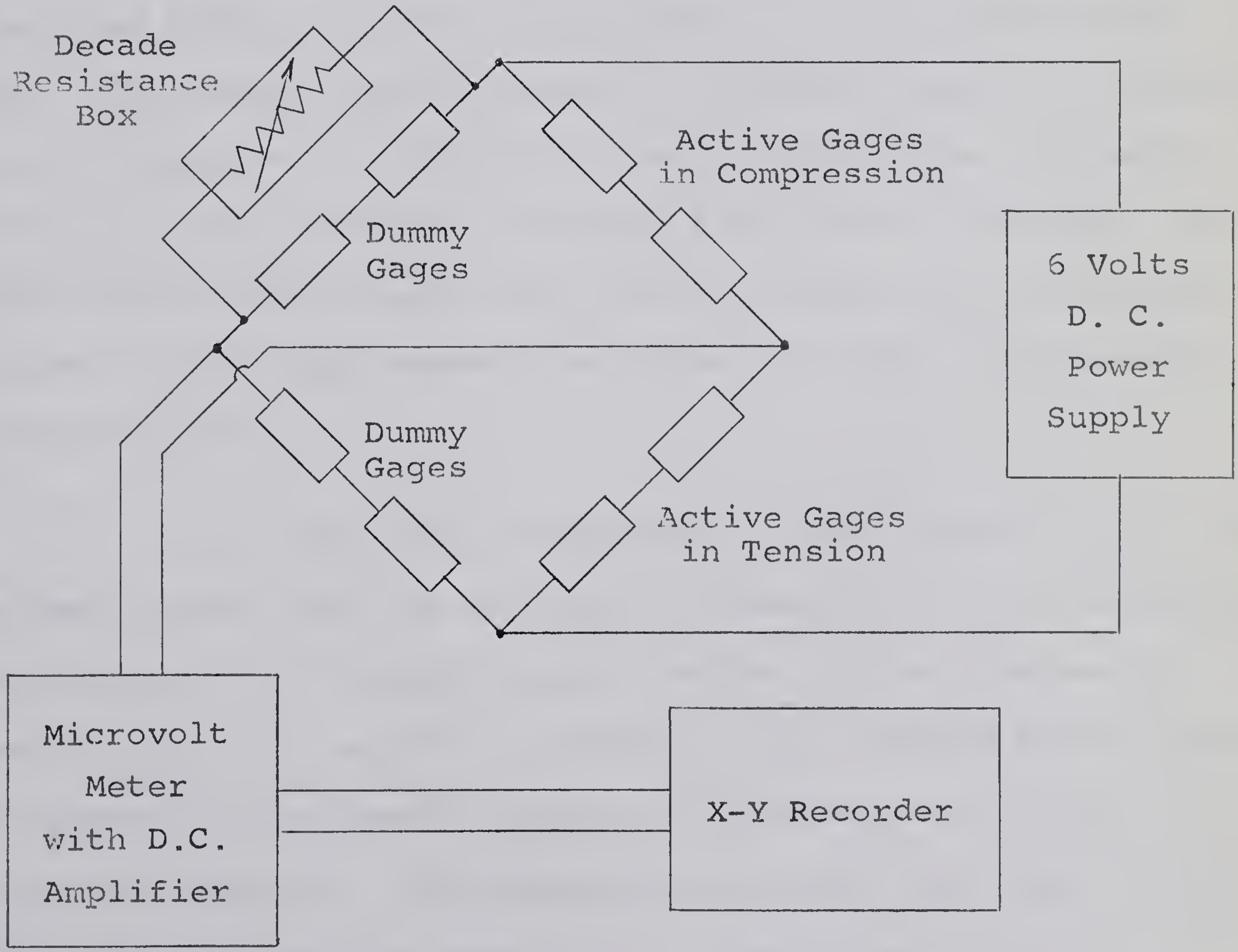


Fig. 5.7 Circuit Diagram for Torque Measurement

gages on the spindle at 45 degrees to the axis were arranged so that two were in compression and two in tension. The changes in resistance of these gages were measured by the bridge-circuit shown in Fig. 5.7 and the bridge was balanced by adjusting the decade resistance box until the microvolt meter read zero. An amplification of the input voltage was taken from a terminal at the back of the microvolt meter and fed into the y-channel of an x-y recorder with the x-channel in time sweep; thus, a continuous record of the torque was obtained during torsion of each specimen. In Fig. 5.2, the microvolt meter, the 6-volt power supply, and the decade resistance box are shown on the foreground table, and the x-y recorder is on the background table. The torque was calibrated by actually applying increments of pure torque on the stationary head in the manner indicated in Fig. 5.5 and recording the signals on the x-y recorder. The length of the arm on each side was 10.0 inches, and the weights applied on each side ranged from 2 lbs. to 7 lbs. in increments of one or $\frac{1}{2}$ lb.

(5) Temperature Measurement - The temperature of the specimen during test was measured by inserting an iron-constantan thermocouple into the stationary shoulder of the specimen as shown in Fig. 5.3, and the temperature was measured by the Leeds and Northrup potentiometer shown in the foreground of Fig. 5.2 (with cover opened). The temperature was not monitored continuously, but was recorded manually at fixed intervals only.

5.3 The Test Specimens

The test specimens were one inch diameter rods with a narrower test section in the middle, as shown in Fig. 5.8. The fillet radii at both ends of the test section were approximately 3/32 inch. The "gage length" of the test section was then defined as the distance between the two points of tangency of the fillets. This definition of the gage length is not exactly correct because the twisting of the material does not start at cross sections through the points of tangency but at cross sections through the fillets. The cross sections at which twisting starts are difficult to determine, so the above definition of gage length was taken as an approximation. The approximation would not affect the results of the experiment provided the definition is taken consistently throughout the experiment; this was done.

The test specimens were made from one-inch diameter rods formed from commercially pure lead billets by inverted extrusion. It is usually assumed that pure lead anneals at room temperature as quickly as it workhardens. However, it was discovered during preliminary tests that this assumption does not appear to be true for commercially pure lead which is 99.998% pure. It was found that a period of about five days was required for the extruded rods to anneal, and that higher annealling temperatures did not shorten the annealling time. The diameter

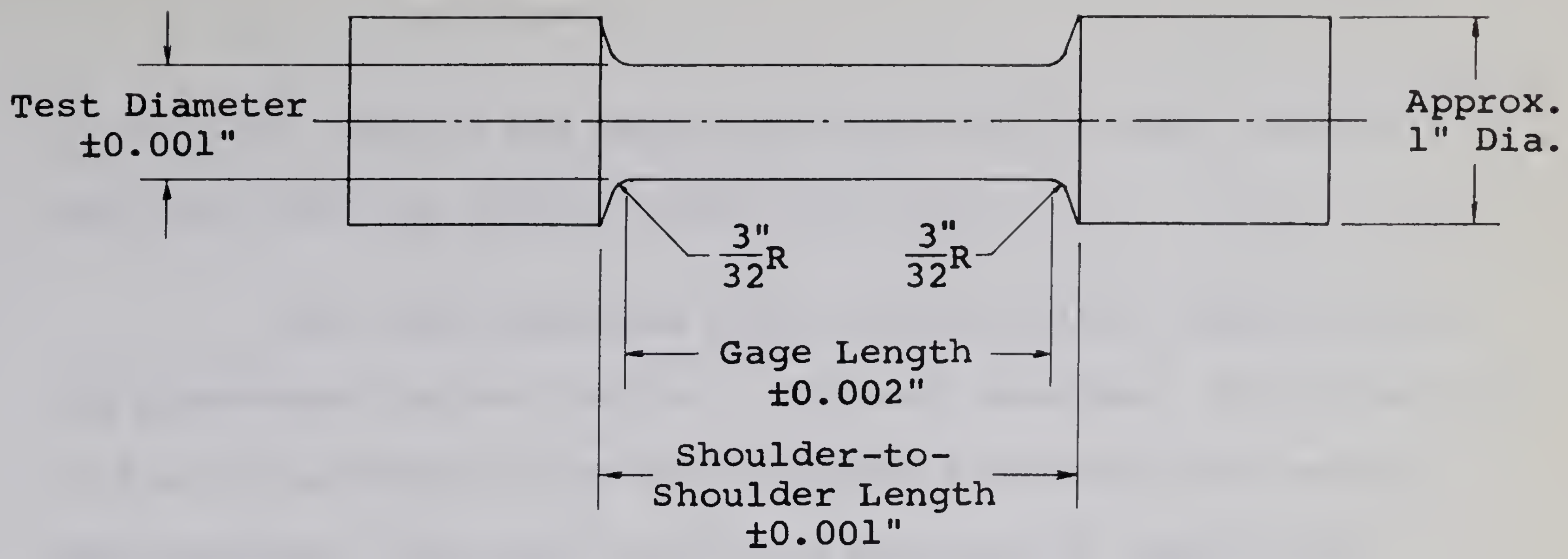
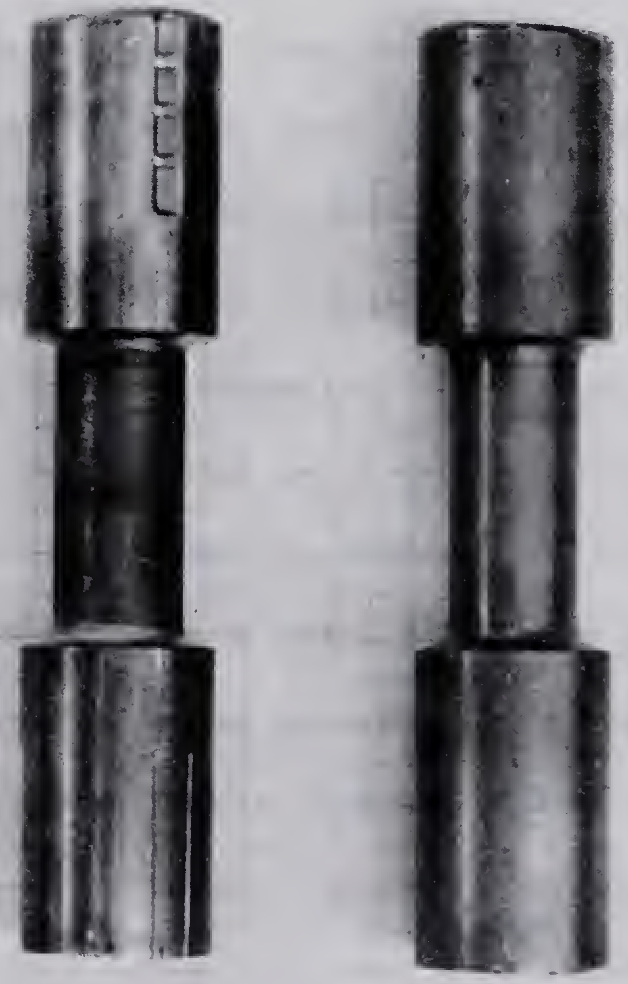


Fig. 5.8 Drawing of a Test Specimen



AFTER TEST

BEFORE TEST

Fig. 5.9 Typical Test Specimens

of the test section was machined to within $\pm$ 0.001" and the taper was kept to within 0.0007" per inch.

All the specimens were measured with a Leitz measuring microscope before testing. For each specimen, the diameters in four directions and at three points along the test section were measured; the gage length was measured by judging the points of tangency of the fillets and four measurements were made around the circumference; two shoulder-to-shoulder length measurements were made around the circumference. In this way, the taper of the test section was checked, the average diameters and the average shoulder-to-shoulder lengths were obtained to within $\pm$ 0.001", and the average gage lengths were obtained to within $\pm$ 0.002". Two typical test specimens are shown in Fig. 5.9.

About 130 5/8-inch diameter specimens were tested. About 70 of these specimens were used for preliminary tests. Preliminary tests were performed to determine the effect of gage length on the torque for a given rate of twist per unit length. It was found that change of gage length had an effect on the torque. At a given rate of twist per unit length, a longer gage length gave a lower torque. This is probably due to end effects of the test section. The material near the non-deforming shoulders does not twist as much as the material away from the shoulders; thus, the gage length is affected. Since it is

difficult to determine quantitatively this end effect, to avoid this complexity, a constant gage length of approximately 1.33" (or approximately 1.5" shoulder-to-shoulder length) was used only.

Three batches of commercially pure lead were tested, and were referred to as Lead A, B, and C. The suppliers and assays are given below:

(1) Lead A - Supplied by Canada Metals Ltd. Its assay obtained by approximate analysis is 0.02% antimony, 0.015% arsenic, and by difference 99.965% lead.

(2) Lead B - Supplied by Consolidated Mining and Smelting Company of Canada, Ltd. The supplier's assay is 0.0002% silver, 0.0002% copper, 0.0003% bismuth, 0.0002% thallium, 0.0001% arsenic, 0.00002% antimony and by difference 99.998% lead.

(3) Lead C - Supplied by Consolidated Mining and Smelting Company of Canada Ltd. The supplier claimed that this batch is the same as Lead B; however, slightly different results were obtained from tests.

One hollow specimen with 3/4-inch outer test diameter and 1/2-inch inner diameter was tested. It was machined from a 1-inch diameter lead tube which was formed by tube extrusion. When twisted, it started to buckle and a "wringing" phenomenon occurred. The same phenomenon was observed by Swift (24) for hollow specimens. The phenomenon may be caused by instability due to slight eccentricity of the hole or slight inhomogeneity of the material.

5.4 Experimental Procedure

The experiments consisted of twisting the specimens at various rates of twist per unit gage length over a range of 0.75 rpm/inch to 11 rpm/inch.

The experimental procedure for testing each specimen started with the calibration of the torque cell by first clamping the calibration arm on the stationary head as shown in Fig. 5.5. Before weights were applied on the arm, the bridge circuit was first balanced, the sweep of the torque measuring x-y recorder was started, and the pen was zeroed. Then increments of weight were applied to give torques of 70, 80, and 90 inch-pounds. After each increment of weight was applied, a slight pressure was applied and released manually on the arm in the direction that increases the torque on the head. This reversed the direction of the impending motion of the pulleys and consequently the direction of the frictional forces. A slight pressure was applied and released again on the arm in the opposite direction returning the pulleys to the original impending motion. The average of the traces obtained from the two impending motions was taken as the true torque calibration. However, friction in the pulleys was not significant and negligible difference between the two traces was noticed. A typical calibration trace is shown in Fig. 5.10. Calibration of the torque cell was done for each specimen tested in order to eliminate the error of possible

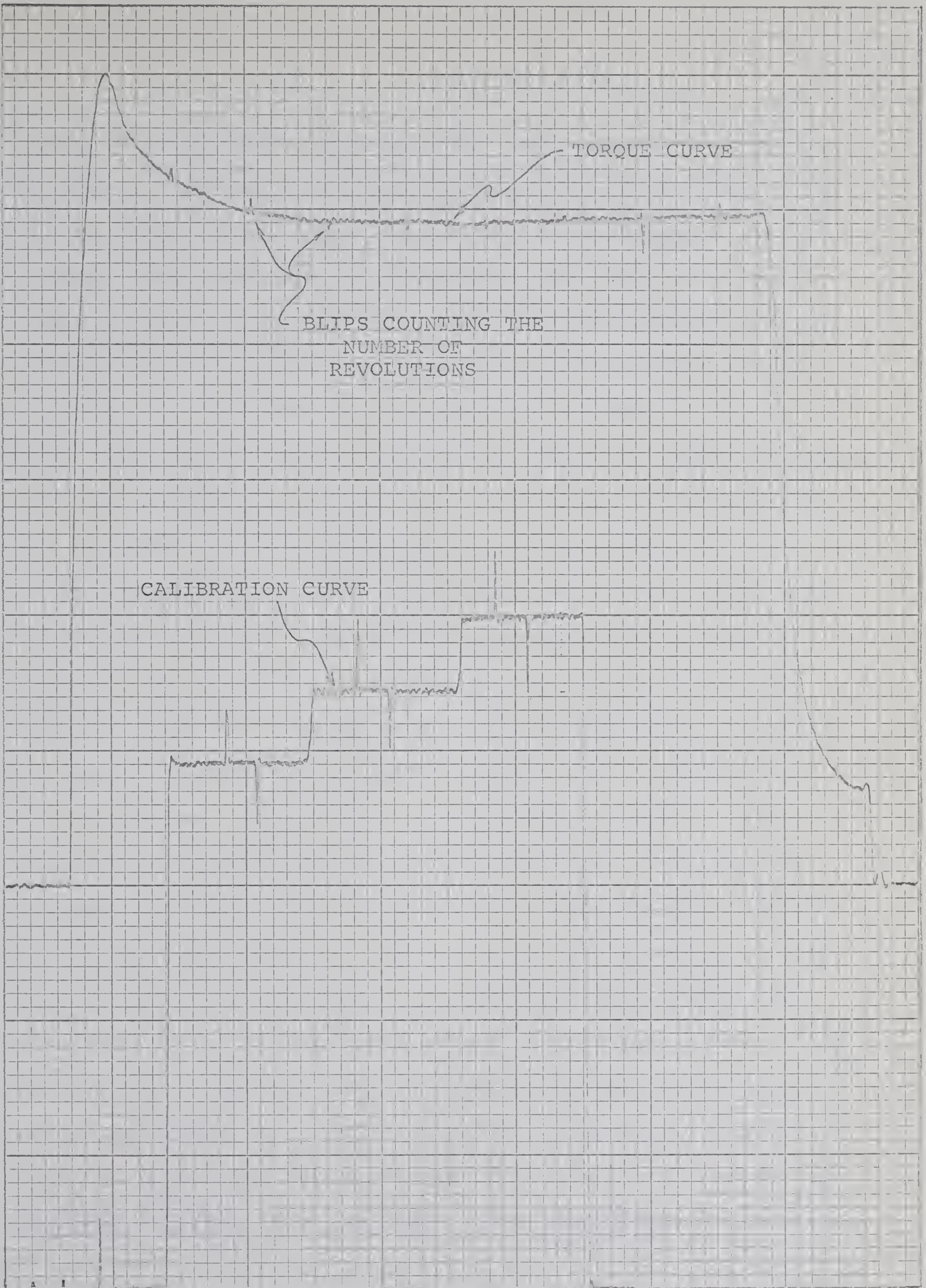


Fig. 5.10 Typical Torque and Calibration Curves

change of calibration between tests; also it was much more convenient to have the calibration on the same chart sheet as the torque curve. After the calibration, the machine was started and the speed was adjusted to the required deflection on the speed monitoring x-y recorder; the required deflection to obtain a given rate of twist during the test was determined by preliminary tests. After the speed adjustment was made, a "no load" trace was made on the x-y recorder to check the speed adjustment. Then the machine was stopped, the calibration arm was removed and replaced by the specimen, and the thermocouple was inserted in the stationary shoulder of the specimen as near as possible to the test section. Before the lower shoulder of the test specimen was clamped in the moving head, the balance of the bridge circuit was checked, the torque measuring x-y recorder was zeroed and started to sweep, and the initial temperature of the specimen was recorded from the potentiometer. Then the specimen was clamped in the moving head making certain that movement clearance was provided in the slider at the coupling of the moving head shaft and that the specimen is not pre-loaded, that is, the torque measuring x-y recorder still reads zero. With both x-y recorders reading zero and sweeping, the machine was started. Then the rate of twist curve and torque curves were simultaneously recorded on the x-y recorders. Also simultaneously, the time for every second revolution of the moving head was obtained from the clock. The time intervals were

recorded manually. As each time the relay coil was energized to start and stop the clock, the magnetic induction from the coil caused a blip on the torque curve; these blips conveniently indicated the number of revolutions but did not affect the accuracy of the torque measurement. At every click of the relay, the temperature was recorded manually, giving the temperature of the specimen at every revolution. (A typical torque trace is shown in Fig. 5.10). While the specimen was being twisted, it was checked constantly for transverse shear failure. If this occurred, the machine was stopped. If no shear failure occurred, the machine was stopped after $1\frac{1}{2}$ to 2 minutes (or 4 to 20 revolutions) of twisting, depending on the speed of twist. After the machine was stopped, the moving head jaws were loosened freeing the torque cell from torque; then the torque trace was checked to see if the pen returned to zero. After the specimens were tested, the shoulder-to-shoulder lengths and the diameters were measured again by the measuring microscope.

5.5 Experimental Results and Discussion

All the torque curves show a bump at the beginning of the test, then the torque drops to a minimum and increases slightly during the rest of the test. No explanation is offered for this initial bump. Such bumps were obtained by

Loizou and Sims (20) and by Kobayashi, et al (21) in compression tests. The slight subsequent increase of torque during the test is due to the increase in diameter of the specimen, which results from the shortening of the specimen. The torque used for correlation with the theoretical result (5.6) was taken as the minimum torque after the bump, that is before any significant shortening and increase in diameter occurred.

The rate of twist was calculated from the clock reading for the revolution in which the minimum torque occurred. The speed monitoring curve was checked to ensure that the speed during that revolution was uniform. The number of revolutions for each test was obtained by counting the blips on the torque curve, and the change of length of each specimen was obtained from the difference between the shoulder-to-shoulder lengths before test and after test. (Continue on page 87).

Table 5.1

Summary of Data for Lead A

Specimen No.	Gage Length L (in.)	Rate of Twist ω (rpm)	$\frac{\omega}{L}$ (rpm/in.)	Torque (in./lbs.)	Change in Temp. (°F) At Min. Torque	At End of test
54	1.353	2.02	1.49	77.8	1.9	2.5
52	1.362	2.06	1.51	78.3	2.7	2.8
53	1.340	2.67	1.99	78.8	3.5	4.4
4	1.309	3.57	2.73	81.5	1.9	5.2
55	1.362	4.55	3.34	82.2	5.4	6.9
21	1.332	5.17	3.88	82.5	5.6	8.0
56	1.378	5.35	3.88	83.1	6.3	8.5
5	1.309	6.00	4.59	85.0	-	-
34	1.338	7.23	5.40	87.0	9.7	12.3
29	1.345	7.31	5.44	89.5	11.8	14.3
33	1.327	7.70	5.80	86.3	9.6	11.8
24	1.333	8.11	6.09	86.9	10.5	13.7
31	1.336	8.82	6.60	88.6	12.1	15.8
23	1.336	8.95	6.70	87.0	10.7	15.5
30	1.331	9.23	6.94	88.3	15.3	18.0
22	1.316	9.53	7.25	86.5	14.0	17.6

Table 5.2

Continuation of Summary of Data for Lead A

Specimen No.	Gage Length Before Test (in.)	Shoulder-to-Shoulder Length (in.) Before Test	Shoulder-to-Shoulder Length (in.) After Test	$\frac{\Delta L}{L}$ (%)	No. of Revs. Twisted	$\frac{ND}{L}$
54	1.353	1.5081	1.4943	1.02	4.60	2.12
52	1.362	1.5083	1.4922	1.18	4.70	2.16
53	1.340	1.5020	1.4811	1.56	8.40	3.92
56	1.378	1.5200	1.4977	1.62	8.70	3.95
55	1.362	1.5108	1.4883	1.65	9.60	4.41
4	1.309	1.5037	1.4832	1.57	12.35	5.90
34	1.338	1.5093	1.4817	2.06	16.50	7.71
5	1.309	1.5003	1.4672	2.53	16.95	8.09
29	1.345	1.5122	1.4752	2.75	17.20	7.99
33	1.327	1.5035	1.4642	2.96	20.85	9.82
31	1.336	1.5088	1.4713	2.81	21.60	10.10
30	1.331	1.5059	1.4626	3.25	23.10	10.85
23	1.336	1.5103	1.4656	3.35	24.05	11.25
22	1.316	1.4999	1.4534	3.53	26.70	12.68

Table 5.3

Summary of Data for Lead B

Specimen No.	Gage Length L (in.)	Rate of Twist ω (rpm)	$\frac{\omega}{L}$ (rpm/in.)	Torque (in./lbs.)	Change in Temp. (°F) At Min. Torque	Change in Temp. (°F) At End of test
72	1.331	2.00	1.50	79.0	2.0	2.9
71	1.345	2.03	1.51	78.0	2.5	2.9
75	1.341	2.64	1.97	78.4	2.8	3.7
85	1.331	3.61	2.71	81.7	3.4	5.1
76	1.352	4.38	3.24	81.2	3.9	6.8
77	1.333	5.05	3.80	82.6	5.6	8.3
78	1.362	6.00	4.40	84.6	6.7	10.7
88	1.381	6.19	4.48	85.0	6.5	9.7
86	1.337	6.67	4.99	84.5	8.0	11.1
79	1.335	6.67	5.00	83.9	6.2	10.8
87	1.335	6.82	5.11	85.5	7.2	10.8
90	1.324	7.06	5.33	86.0	8.1	12.3
92	1.359	8.00	5.89	85.8	9.6	11.8
96	1.347	8.82	6.55	83.9	8.2	12.5
69	1.343	10.00	7.44	84.7	9.6	17.7
95	1.346	11.32	8.42	86.0	8.8	18.0
73	1.362	12.50	9.19	86.0	12.3	22.5
97	1.331	12.24	9.20	86.3	9.3	15.6
93	1.365	13.64	10.00	85.8	11.3	18.7
98	1.326	15.00	10.92	86.0	12.3	21.0
94	1.373	15.40	11.20	87.3	12.7	25.7

Table 5.4

Continuation of Summary of Data for Lead B

Specimen No.	Gage Length Before Test (in.)	Shoulder-to-Shoulder Length (in.) Before Test	After Test	$\frac{\Delta L}{L}$ (%)	No. of Revs. Twisted	$\frac{ND}{L}$
72	1.331	1.4966	1.4820	1.10	4.45	2.09
71	1.345	1.5009	1.4884	0.93	4.65	2.16
75	1.341	1.5041	1.4851	1.42	6.15	2.87
85	1.331	1.5016	1.4834	1.37	6.25	2.93
89	1.358	1.5203	1.5037	1.22	6.40	2.95
76	1.352	1.5132	1.4915	1.60	8.85	4.09
88	1.381	1.5476	1.5232	1.74	10.00	4.53
77	1.333	1.4962	1.4650	2.34	12.35	5.79
86	1.337	1.5020	1.4700	2.39	12.80	5.98
78	1.362	1.5237	1.4871	2.69	13.25	6.08
90	1.324	1.4882	1.4479	3.04	13.60	6.42
92	1.359	1.5146	1.4746	2.94	14.45	6.65
79	1.335	1.5060	1.4652	3.06	14.70	6.88

Table 5.5

Summary of Data for Lead C

Specimen No.	Gage Length L (in.)	Rate of Twist ω (rpm)	$\frac{\omega}{L}$ (rpm/in.)	Torque (in./lbs.)	Change in Temp. (°F) At Min. Torque	At End of test
102	1.346	0.985	0.73	74.0	1.6	1.6
106	1.352	1.004	0.74	75.3	1.5	1.5
103	1.329	1.95	1.47	76.8	2.7	3.0
104	1.321	1.97	1.49	78.0	2.4	3.0
111	1.353	3.06	2.26	82.5	4.0	4.7
110	1.372	3.11	2.27	80.5	4.2	5.5
113	1.313	4.14	3.15	83.0	4.8	6.5
112	1.337	4.29	3.20	83.2	5.0	6.8
119	1.355	5.09	3.75	85.7	5.5	7.5
114	1.348	5.13	3.80	83.8	5.5	8.8
120	1.346	5.94	4.41	87.2	7.1	9.3
124	1.332	6.12	4.60	86.1	6.9	9.3
125	1.367	6.98	5.10	86.4	5.7	11.7
118	1.345	7.32	5.45	85.8	-	-
126	1.385	7.90	5.70	87.0	7.3	11.0
100	1.340	8.95	6.69	88.2	9.8	14.3
117	1.353	9.38	6.93	87.6	8.7	11.4
107	1.335	10.34	7.75	87.2	8.7	11.6
108	1.329	10.53	7.93	88.5	9.1	14.5
101	1.337	11.54	8.64	89.2	11.7	17.5
121	1.356	12.00	8.85	88.9	12.3	17.3
115	1.363	13.33	9.78	89.4	9.9	15.1
116	1.332	13.05	9.80	89.1	9.6	16.0
123	1.377	15.00	10.90	89.4	-	-
122	1.358	15.00	11.03	87.6	12.0	19.6

Table 5.6

Continuation of Summary of Data for Lead C

Specimen No.	Gage Length Before Test (in.)	Shoulder-to-Shoulder Length (in.) Before Test	Shoulder-to-Shoulder Length (in.) After Test	$\frac{\Delta L}{L}$ (%)	No. of Revs. Twisted	$\frac{ND}{L}$
102	1.346	1.5108	1.4980	0.95	3.5	1.63
106	1.352	1.5076	1.4934	1.05	4.0	1.85
104	1.321	1.4980	1.4847	1.01	4.4	2.08
103	1.329	1.5085	1.4925	1.20	4.5	2.12
110	1.372	1.5145	1.4942	1.48	6.2	2.82
113	1.313	1.4826	1.4660	1.26	6.3	3.00
111	1.353	1.5018	1.4863	1.15	6.7	3.10
119	1.355	1.5094	1.4824	1.99	6.9	3.18
117	1.353	1.5084	1.4740	2.54	7.1	3.28
107	1.335	1.4951	1.4673	2.08	7.2	3.37
120	1.346	1.4998	1.4615	2.85	8.1	3.76
124	1.332	1.5044	1.4516	3.96	8.6	4.04
109	1.335	1.5155	1.4742	3.09	8.7	4.07
118	1.345	1.5027	1.4627	2.97	8.9	4.14
108	1.329	1.4959	1.4579	2.86	8.9	4.19
115	1.363	1.5262	1.4651	4.48	9.6	4.40
121	1.356	1.5141	1.4468	4.96	10.7	4.93
125	1.367	1.5257	1.4600	4.80	10.9	4.98
116	1.332	1.4949	1.4260	5.17	10.8	5.07
100	1.340	1.5082	1.4525	4.16	11.1	5.18
122	1.358	1.5136	1.4478	4.85	11.5	5.29
101	1.337	1.5089	1.4398	5.17	12.7	5.94

Table 5.7

Estimated Accuracy of Data

Data	Accuracy
Gage Length	± 0.002"
Rate of Twist	± 1.5%
Torque	± 0.3 in-lbs.
No. of Revs.	± 0.5%
Change of Length	± 0.003"

For specimens that were twisted at rates of twist per unit gage length from approximately 5.0 rpm/inch to 5.5 rpm/inch, the torque curves were wavy, as shown in Fig.5.11. The possibility that this was due to a machine characteristic was investigated and eliminated, and no satisfactory explanation could be found for this irregularity. Accurate torques could not be obtained for this range from the torque curves because of the waviness, so only the average torques were taken as approximations in the range 5.0 rpm/inch to 5.5 rpm/inch.

All the data obtained for each specimen are presented in Tables 5.1 to 5.6. The estimated accuracy of the data are given in Table 5.7. The curves plotting torques against rates of twist per unit length $(\frac{\omega}{L})$ and plotting percentage shortening of the original gage length against the dimensionless parameter $\frac{ND}{L}$, where D is the diameter of the specimen and L is the

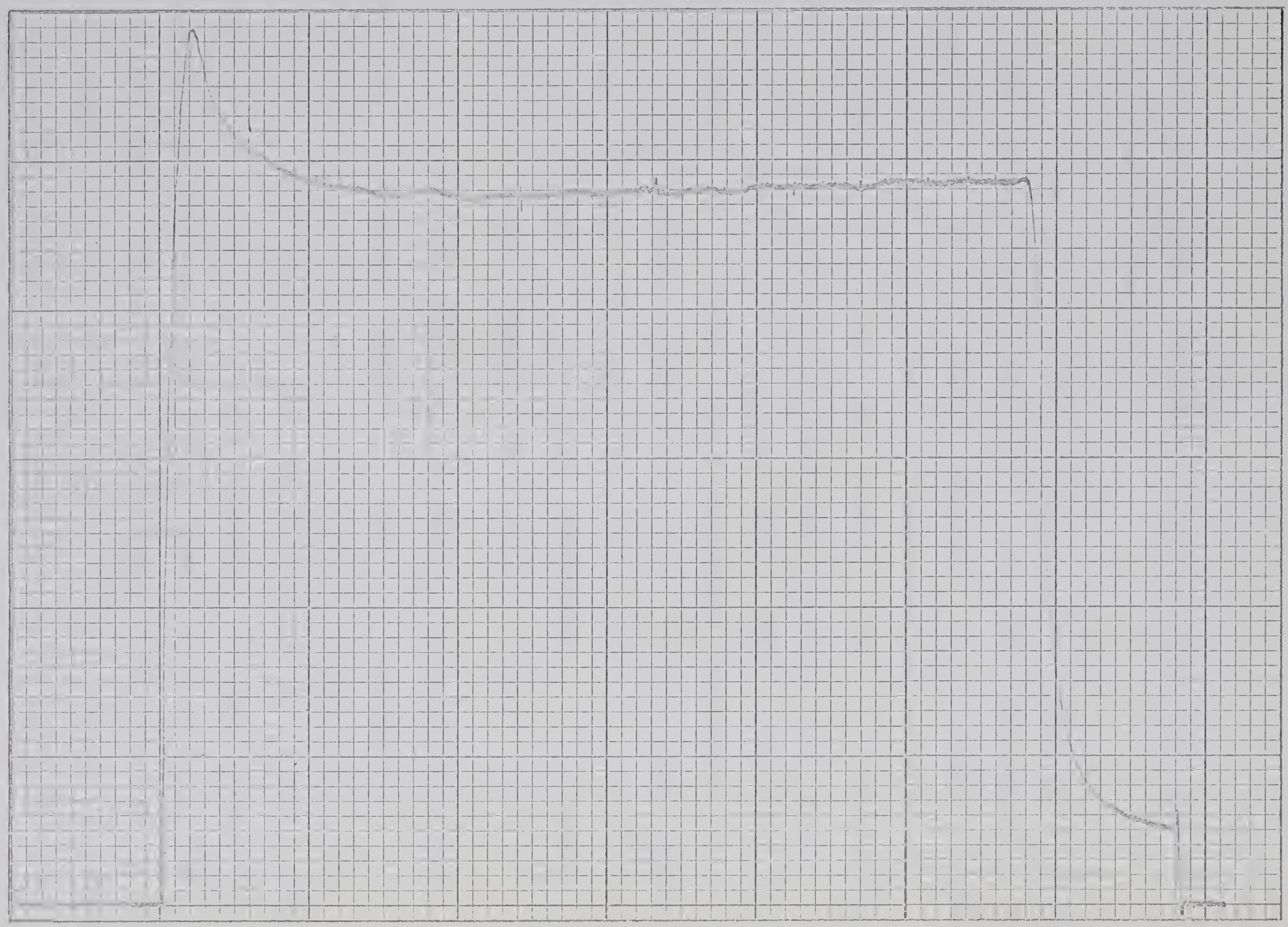


Fig. 5.11 A Typical Wavy Torque Curve

original gage length, are shown in Figs. 5.12 to 5.17 for the three batches of pure lead.

The torques appear to have a linear relation with rates of twist per unit length for $\frac{\omega}{L} < 5.0$ rpm/inch , and appear to be independent of $\frac{\omega}{L}$ for $\frac{\omega}{L} > 5.5$ rpm/inch. In the region between 5.0 rpm/inch and 5.5 rpm/inch, the data are quite scattered; consequently, no conclusion can be made. There is a possibility that a critical region exists near the point where the rate dependent characteristics of lead change. This region corresponds approximately to a maximum shear strain rate (engineering definition) in the specimen of about 0.16 sec^{-1}. In the investigation of Loizou and Sims (20) using the compression test, no such critical region existed. This critical phenomenon may be inherent to the torsion test, which subjects the material to shear only.

Straight lines were fitted by the method of least squares to the torque versus $\frac{\omega}{L}$ curves for the range of $\frac{\omega}{L} < 5.0$ rpm/inch. For each lead, equation (5.16) in the form $T = A_o + A_1 (\frac{\omega}{L})$ was obtained. The constants A_o and A_1, are shown in Table 5.8. From these constants, the coefficients of viscosity η and yield stresses in shear k were calculated, and are also shown in Table 5.8. An estimate of the error of η may be obtained by an estimate of the error of A_1, which is the slope of the fitted straight line. From a statistics reference (26),

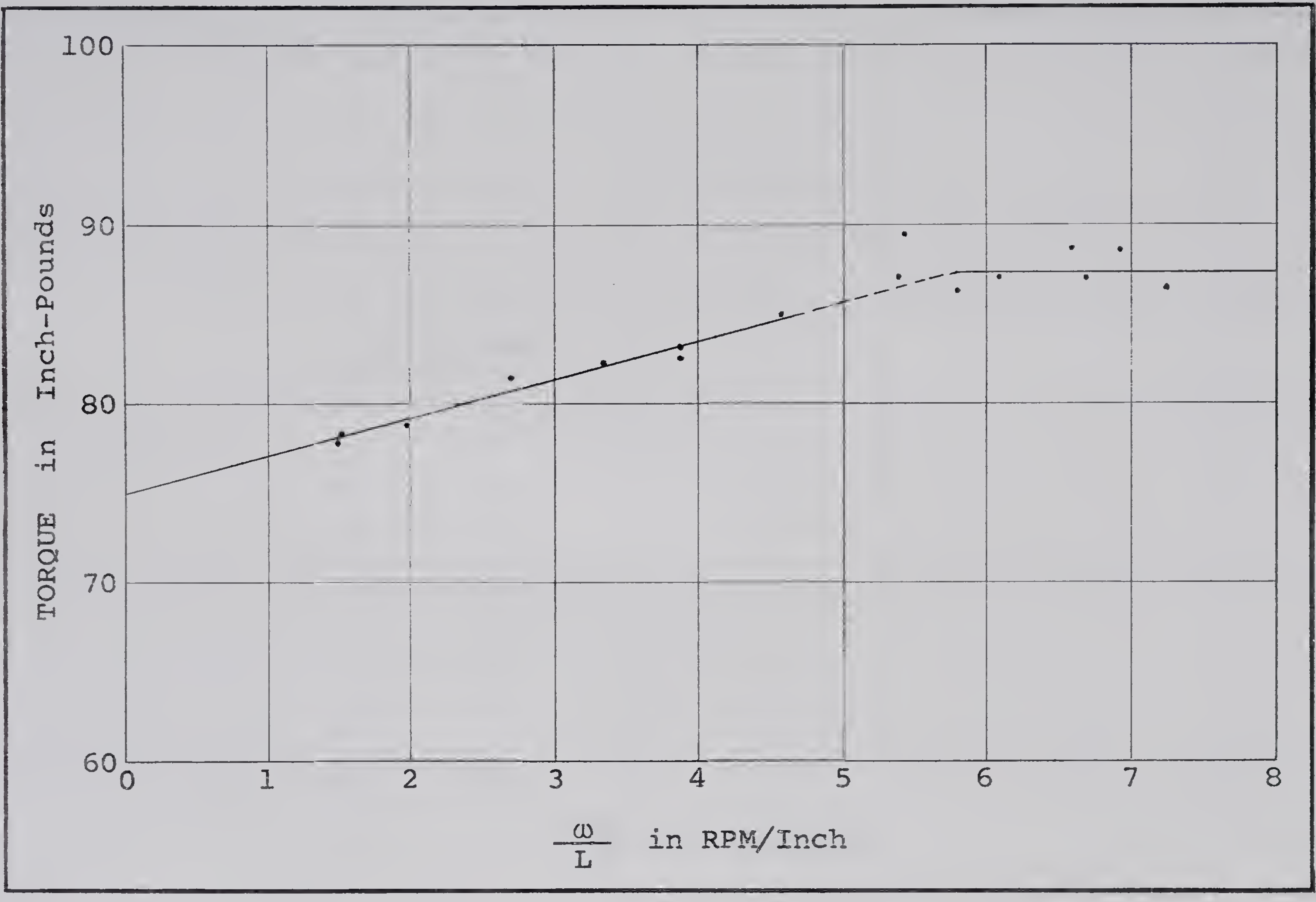


Fig. 5.12 Torque vs. Rate of twist per Unit Length for Lead A

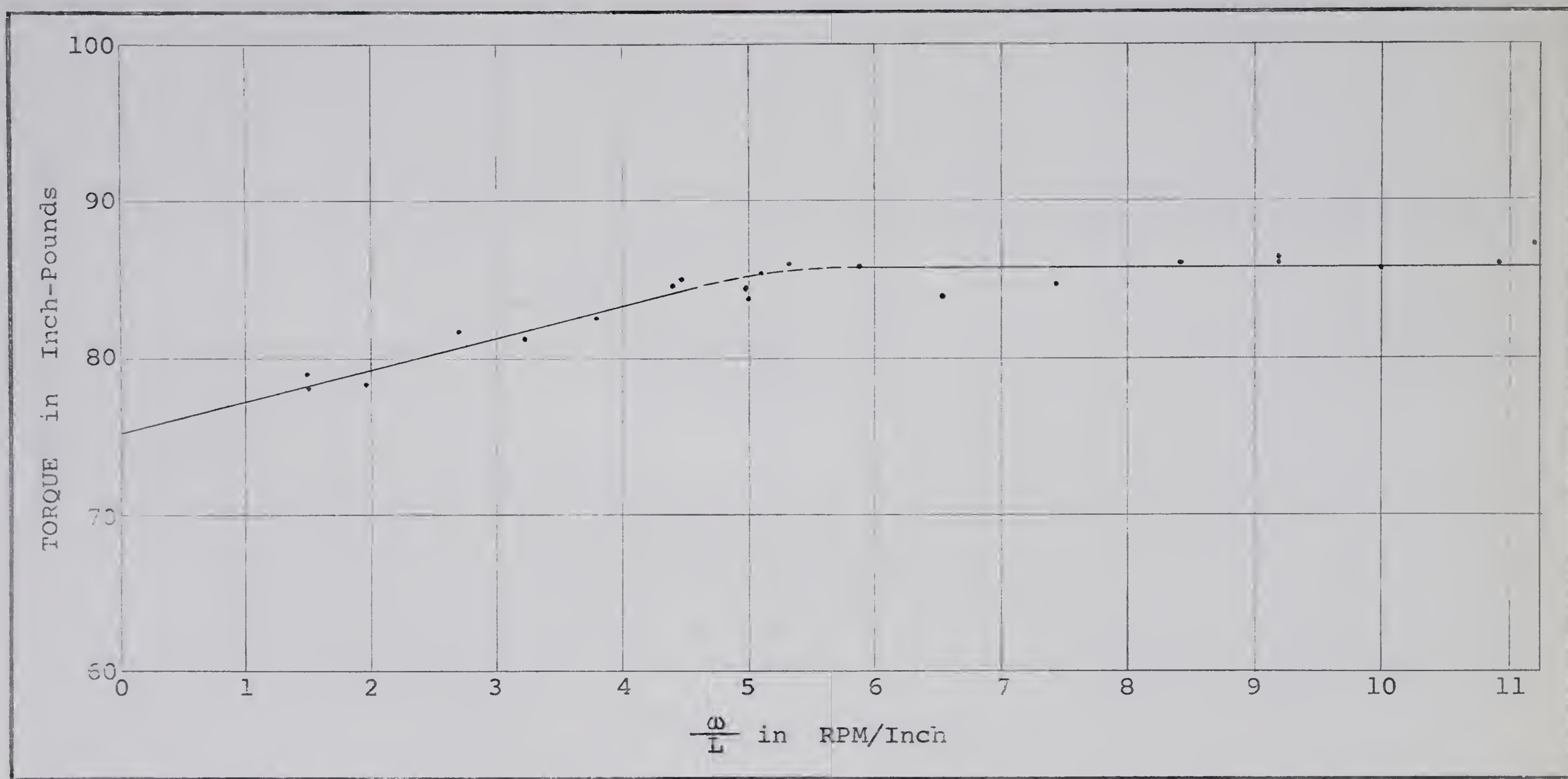


Fig. 5 13 Torque vs. Rate of Twist per Unit Length for Lead B

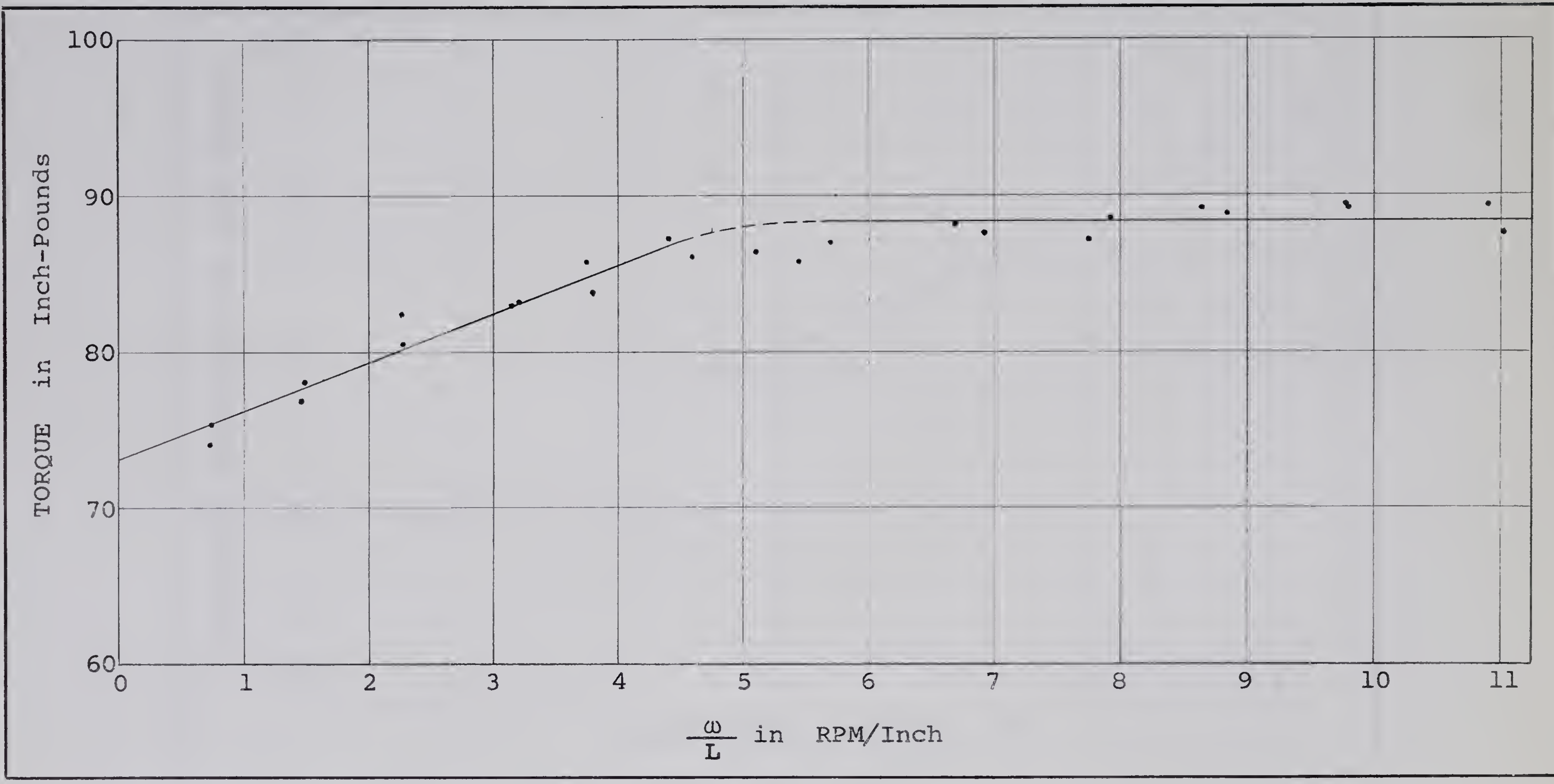


Fig. 5.14 Torque vs. Rate of Twist per Unit Length for Lead C

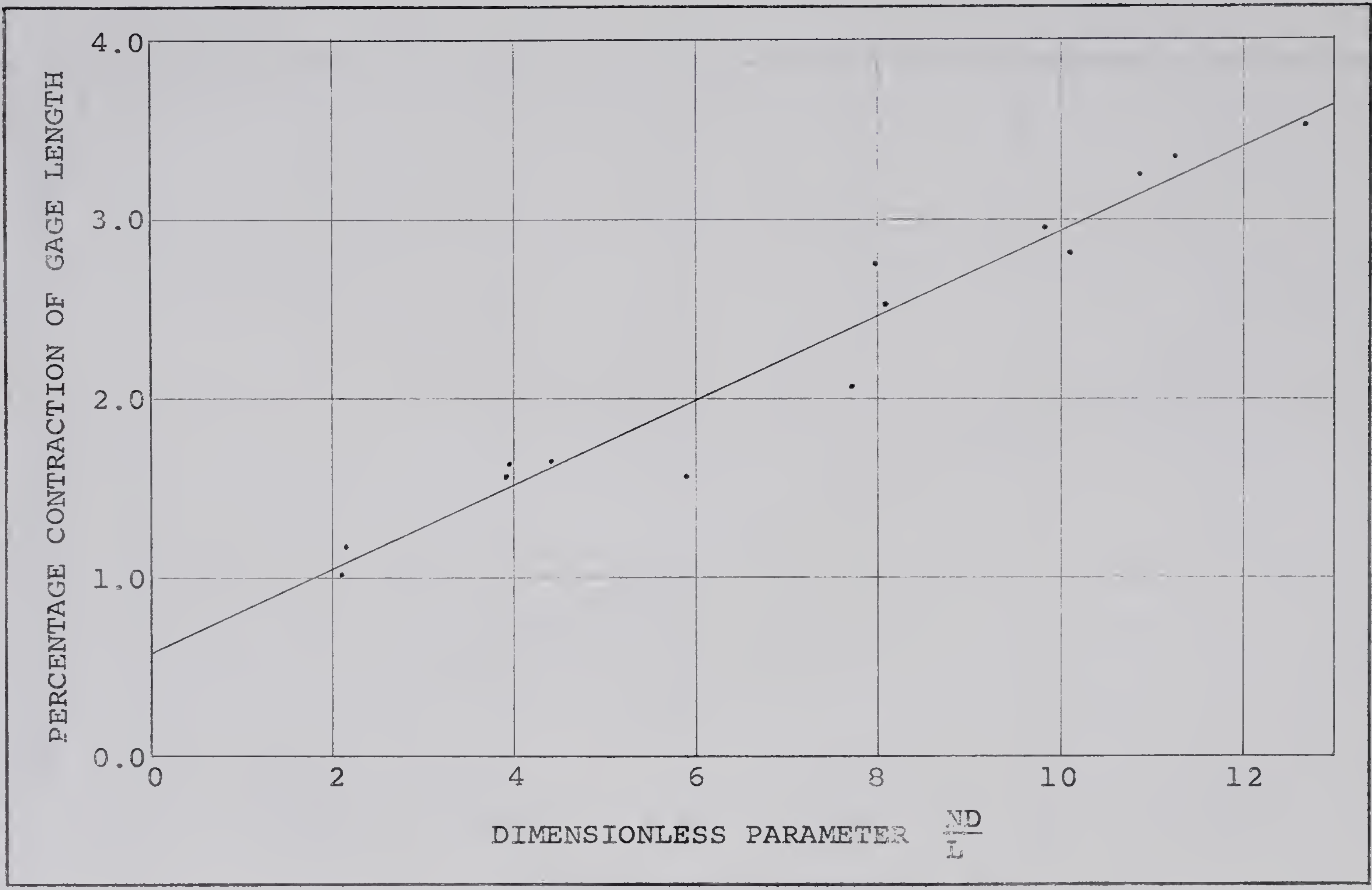


Fig. 5.15 Percentage Contraction of Gage Length vs. Dimensionless Parameter ND/L for Lead A

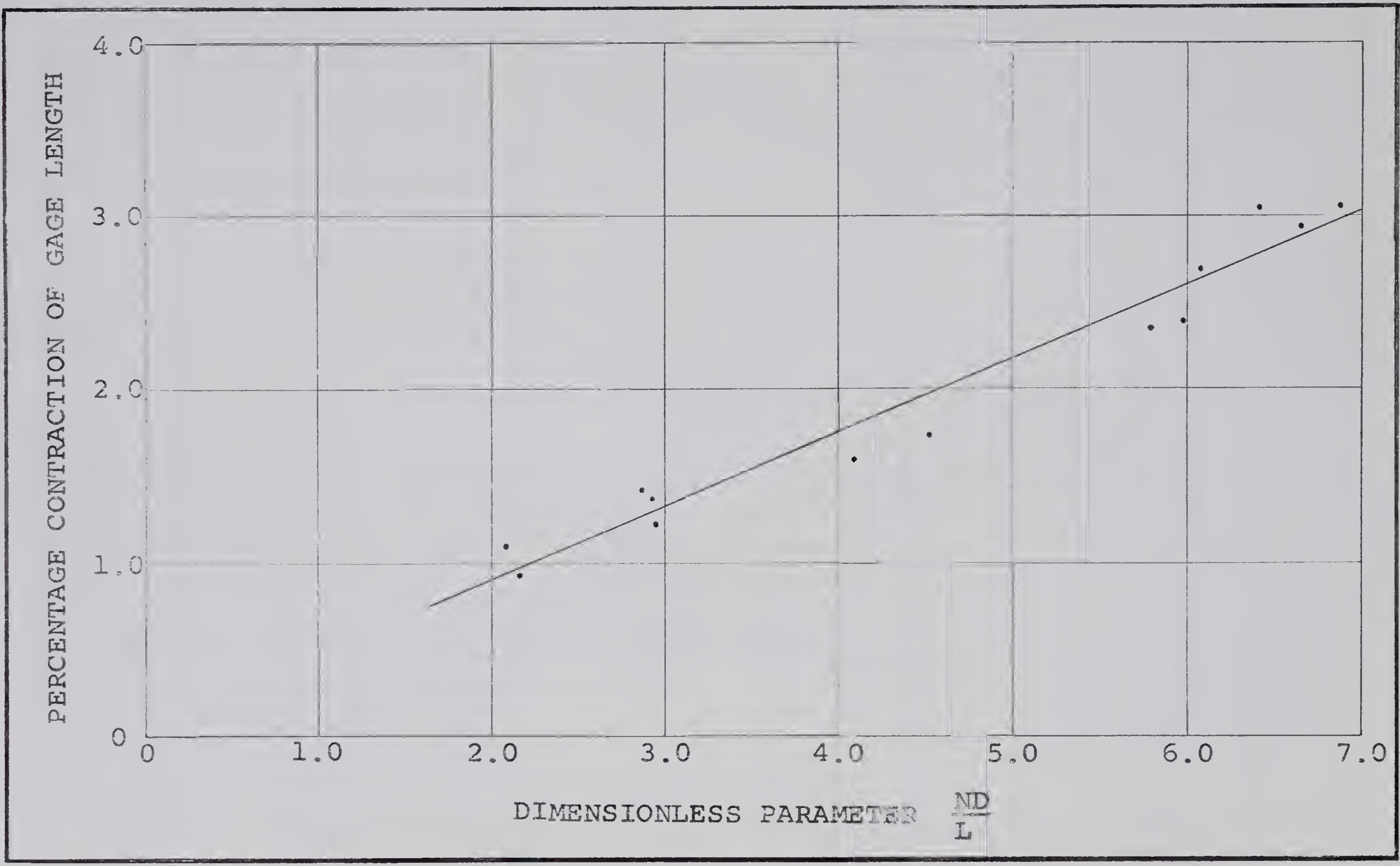


Fig. 5.16 Percentage Contraction of Gage Length vs. Dimensionless Parameter ND/L for Lead B

Fig. 5.17 Percentage Contraction of Gage Length vs. Dimensionless Parameter ND/L for Lead C

the standard errors of the slopes were calculated. Since the error of the radius of the specimen is negligible, the percentage error of slope would then be the percentage error of η. (Refer to equation (5.6)). The estimated percentage errors of η for the three batches of lead are shown also in Table 5.8.

Table 5.8

Coefficients of Viscosity and Yield Stresses in Shear

	A_o	A_1	η (psi-sec)	k (psi)	Estimated % Error of η
Lead A	74.85	2.152	1372	1170	7%
Lead B	75.19	2.047	1305	1176	13%
Lead C	73.02	3.135	1998	1142	8%

It can be seen from Table 5.8 that the rate sensitivity of commercially pure lead varies greatly with small changes of composition. This contributed to the difficulties of the experiments. The variation of the rate sensitivity may be due to the extent of annealling of the lead, the casting temperature, or the rate of cooling when the lead was cast.

The data given in Figs. 5.15 and 5.16 show a linear correlation between the percentage shortening and the parameter $\frac{ND}{L}$, so straight lines were fitted to the data. In Fig. 5.17, a best fit straight line was drawn for the data, but the scattering is

such that no definite conclusion can be made. Irregularity due to friction at the slider coupling of the shaft of the moving head is a possible reason for the scattering of the data. Swift (24) twisted one lead specimen; the curve plotting percentage shortening against shear strain at the outer diameter during torsion showed also a linear relationship.

The slopes of the straight lines fitted to data in Figs. 5.15 to 5.17 are respectively 0.237, 0.424, and 1.204. This variation of the slopes shows that the shortening appears to be quite sensitive to the metallurgy of the pure leads.

Swift (24) gave an explanation for the change of length of material during pure torsion by qualitative considerations of the differential grain orientations. Hill (25) suggested that the change of length is due to both anisotropy and differential orientations of the grains, and derived an equation for the change of length assuming the change is solely due to anisotropy.

5.6 Conclusions and Recommendations

The results of this experimental investigation show that the torsion test can be used to assess strain rate sensitivity of materials, although the change in length of the specimen and the bump in the torque curve may influence the assessment. The linear dependence of the torque on the rate of twist per unit length

shows that commercially pure lead behaves according to the Bingham visco-plastic model; however, this is true only under a certain critical region; beyond the critical region, the torque is independent of rate of twist per unit length. The results show that reasonable coefficients of viscosity can be obtained by the torsion test.

The critical region is of interest; however, no general conclusion can be made and further work is needed to investigate this phenomenon. Further work may also be done by varying the radius of the specimen to verify further that pure lead behaves according to the Bingham visco-plastic model. For future work, better metallurgical control of the pure lead is suggested.

The correlation between the shortening of the specimens and the parameter proportional to the number of revolutions twisted shows a tendency to be linear; however, because of the scattering of the data, no definite conclusion can be made. Further work may be done by improving the method of allowing the specimen to change length during twist and the method of measuring the change of length, for example, through continuous measurement by electrical displacement transducers.

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APPENDIX

COMPUTER PROGRAMS AND CALCULATED RESULTS

The computer programs are written in Fortran IV Language for an IBM 7040 Digital Computer. The following nomenclatures are used throughout all the computer programs and symbols not listed are particular to each program:

E	=	ϵ
H	=	grid width
N	=	one plus the number of grid divisions
P(I,J)	=	$\phi_{(i,j)}$
SI	=	I
SX	=	s_{xz}
SY	=	s_{yz}
ST	=	τ (for Program B)
T(I,J)	=	τ (except Program B)
TQQ	=	T

COMPUTER PROGRAM A

```
      DIMENSION P(51,51),T(51,51),TDX(51,51),TDY(51,51)
      READ (5,100) E,H,N
      NN=N-1
      HH=H*H
      H2=HH/2.
      H6=H/6.
      DO 5 I=1,N
    5 P(I,N)=0.0
      Y=1.0
      J=N+1
   10 J=J-1
      K=J-1
      T(J,J)=1.0
      TDX(J,J)=-2.*E
      TDY(J,J)= 2.*E
      X=-H
      DO 20 I=1,K
      X=X+H
      T(I,J)=1.+2.*E*(Y-X)
      TDX(I,J)=-2.*E
      TDY(I,J)= 2.*E
      TA=SIN(P(I,J))/COS(P(I,J))
   20 P(I,J-1)=P(I,J)+TA*(P(I+1,J)-P(I,J))-H*(TDX(I,J)-TA*TDY(I,J))/T(I,
     1J)
      Y=Y-H
      IF (J.GT.2) GO TO 10
      WRITE (6,101)
      TQ=(1.+E*H/2.)*HH*H/32.
      Y=0.
      DO 45 J=2,NN
      Y=Y+H
      X=0.
      I=1
      SX=T(I,J)*COS(P(I,J))
      SY=T(I,J)*SIN(P(I,J))
      TQ=TQ+(SX*Y+SY*H/4.)*H2
      WRITE (6,102) X,Y,P(I,J),SX,SY,T(I,J)
   25 IF (J.EQ.2) GO TO 42
      K=J-1
      DO 40 I=2,K
      X=X+H
      SX=T(I,J)*COS(P(I,J))
      SY=T(I,J)*SIN(P(I,J))
      TQ=TQ+(SX*Y+SY*X)*HH
   40 WRITE (6,102) X,Y,P(I,J),SX,SY,T(I,J)
   42 I=J
      X=X+H
      SX=T(I,J)*COS(P(I,J))
      SY=T(I,J)*SIN(P(I,J))
      SXU=SX*5./6.+T(I,J+1)*COS(P(I,J+1))/6.
      TQ=TQ+(SXU*(Y+H6)+SY*(X-H6))*H2
   45 WRITE (6,102) X,Y,P(I,J),SX,SY,T(I,J)
      J=N
```

```
      Y=1.0
      YY=1.-H/4.
      X=-H
      DO 50 I=1,NN
      X=X+H
      SX=T(I,J)
      SY=0.
      AA=H2
      IF (I.EQ.1) AA=H2/2.
      TQ=TQ+SX*AA*YY
   50 WRITE (6,102) X,Y,P(I,J),SX,SY,T(I,J)
      TQ=TQ+HH*YY/8.
      TQQ=TQ*8.
      WRITE (6,103) TQQ
      WRITE (6,104) TQ
   80 STOP
  100 FORMAT (F10.3,F9.3,I6)
  101 FORMAT (75H      I     J      X         Y         P(I,J)         T(I,J)
     1     SX              SY/)
  102 FORMAT (11X,2F8.3,4F13.8)
  103 FORMAT (31HK         DIMENSIONLESS TORQUE =,F12.8)
  104 FORMAT (31H          ONE-EIGHTH OF IT     =,F12.8)
      END
```

DATA CARD -

```
   0.100    0.020    51
```

Table A.1

RESULTS FROM PROGRAM A

x	y	ϕ	$-s_{xz}$	s_{yz}	τ
0.000	0.040	0.19189749	0.98949722	0.19224766	1.00799997
0.040	0.040	0.19346200	0.98134452	0.19225745	1.00000000
0.000	0.080	0.18225625	0.99917221	0.18414889	1.01599997
0.040	0.080	0.18372708	0.99103492	0.18415675	1.00799997
0.080	0.080	0.18522210	0.98289536	0.18416484	1.00000000
0.000	0.120	0.17279582	1.00875045	0.17606368	1.02399995
0.040	0.120	0.17417647	1.00062745	0.17606987	1.01599997
0.080	0.120	0.17557956	0.99250243	0.17607624	1.00799997
0.120	0.120	0.17700567	0.98437535	0.17608282	1.00000000
0.000	0.160	0.16350959	1.01823520	0.16799100	1.03199996
0.040	0.160	0.16480335	1.01012541	0.16799575	1.02399997
0.080	0.160	0.16611791	1.00201383	0.16800063	1.01599997
0.120	0.160	0.16745380	0.99390040	0.16800568	1.00799997
0.160	0.160	0.16881154	0.98578513	0.16801090	1.00000000
0.000	0.200	0.15439133	1.02762947	0.15992984	1.03999996
0.040	0.200	0.15560132	1.01953186	0.15993335	1.03199996
0.080	0.200	0.15683055	1.01143269	0.15993696	1.02399997
0.120	0.200	0.15807952	1.00333188	0.15994071	1.01599997
0.160	0.200	0.15934869	0.99522945	0.15994458	1.00799997
0.200	0.200	0.16063857	0.98712534	0.15994858	1.00000000
0.000	0.240	0.14543512	1.03693614	0.15187926	1.04799996
0.040	0.240	0.14656428	1.02884976	0.15188172	1.03999996
0.080	0.240	0.14771123	1.02076201	0.15188426	1.03199996
0.120	0.240	0.14887639	1.01267285	0.15188689	1.02399997
0.160	0.240	0.15006021	1.00458222	0.15188963	1.01599997
0.200	0.240	0.15126314	0.99649014	0.15189246	1.00799997
0.240	0.240	0.15248564	0.98839657	0.15189539	1.00000000
0.000	0.280	0.13663535	1.04615792	0.14383839	1.05599996
0.040	0.280	0.13768650	1.03808186	0.14383996	1.04799996
0.080	0.280	0.13875403	1.03000460	0.14384159	1.03999996
0.120	0.280	0.13983835	1.02192613	0.14384330	1.03199997
0.160	0.280	0.14093985	1.01384639	0.14384507	1.02399997
0.200	0.280	0.14205895	1.00576537	0.14384692	1.01599997
0.240	0.280	0.14319607	0.99768303	0.14384885	1.00799997
0.280	0.280	0.14435165	0.98959937	0.14385086	1.00000000
0.000	0.320	0.12798673	1.05529735	0.13580640	1.06399996
0.040	0.320	0.12896251	1.04723078	0.13580723	1.05599996
0.080	0.320	0.12995336	1.03916314	0.13580811	1.04799996
0.120	0.320	0.13095964	1.03109447	0.13580904	1.03999997
0.160	0.320	0.13198170	1.02302470	0.13581002	1.03199997
0.200	0.320	0.13301992	1.01495383	0.13581105	1.02399997
0.240	0.320	0.13407470	1.00688181	0.13581214	1.01599997
0.280	0.320	0.13514643	0.99880862	0.13581328	1.00799997
0.320	0.320	0.13623553	0.99073427	0.13581449	1.00000000
0.000	0.360	0.11948419	1.06435686	0.12778250	1.07199996
0.040	0.360	0.12038714	1.05629895	0.12778273	1.06399996
0.080	0.360	0.12130390	1.04824014	0.12778300	1.05599996
0.120	0.360	0.12223479	1.04018044	0.12778329	1.04799997
0.160	0.360	0.12318015	1.03211979	0.12778363	1.03999997
0.200	0.360	0.12414031	1.02405819	0.12778399	1.03199997
0.240	0.360	0.12511562	1.01599560	0.12778440	1.02399997
0.280	0.360	0.12610646	1.00793202	0.12778484	1.01599997

0.320	0.360	0.12711318	0.99986741	0.12778531	1.00799997
0.360	0.360	0.12813620	0.99180178	0.12778584	1.00000000
0.000	0.400	0.11112296	1.07333873	0.11976595	1.07999996
0.040	0.400	0.11195548	1.06528872	0.11976571	1.07199996
0.080	0.400	0.11280061	1.05723796	0.11976548	1.06399996
0.120	0.400	0.11365864	1.04918645	0.11976527	1.05599997
0.160	0.400	0.11452988	1.04113411	0.11976509	1.04799997
0.200	0.400	0.11541464	1.03308096	0.11976492	1.03999997
0.240	0.400	0.11631322	1.02502697	0.11976477	1.03199997
0.280	0.400	0.11722597	1.01697215	0.11976465	1.02399997
0.320	0.400	0.11815321	1.00891643	0.11976455	1.01599997
0.360	0.400	0.11909531	1.00085982	0.11976448	1.00799997
0.400	0.400	0.12005261	0.99280232	0.11976444	1.00000000
0.000	0.440	0.10289846	1.08224511	0.11175607	1.08799996
0.040	0.440	0.10366283	1.07420231	0.11175545	1.07999996
0.080	0.440	0.10443868	1.06615887	0.11175484	1.07199996
0.120	0.440	0.10522626	1.05811479	0.11175424	1.06399998
0.160	0.440	0.10602586	1.05007001	0.11175365	1.05599997
0.200	0.440	0.10683774	1.04202455	0.11175306	1.04799997
0.240	0.440	0.10766219	1.03397838	0.11175249	1.03999997
0.280	0.440	0.10849951	1.02593149	0.11175193	1.03199997
0.320	0.440	0.10935000	1.01788386	0.11175138	1.02399997
0.360	0.440	0.11021399	1.00983546	0.11175084	1.01599997
0.400	0.440	0.11109178	1.00178630	0.11175032	1.00799997
0.440	0.440	0.11198372	0.99373637	0.11174982	1.00000000
0.000	0.480	0.09480636	1.09107808	0.10375218	1.09599997
0.040	0.480	0.09550476	1.08304181	0.10375128	1.08799996
0.080	0.480	0.09621355	1.07500500	0.10375039	1.07999996
0.120	0.480	0.09693297	1.06696765	0.10374949	1.07199998
0.160	0.480	0.09766327	1.05892972	0.10374860	1.06399998
0.200	0.480	0.09840468	1.05089120	0.10374771	1.05599997
0.240	0.480	0.09915747	1.04285211	0.10374682	1.04799997
0.280	0.480	0.09992190	1.03481240	0.10374593	1.03999997
0.320	0.480	0.10069823	1.02677206	0.10374503	1.03199997
0.360	0.480	0.10148677	1.01873110	0.10374415	1.02399997
0.400	0.480	0.10228780	1.01068949	0.10374327	1.01599997
0.440	0.480	0.10310161	1.00264722	0.10374239	1.00799997
0.480	0.480	0.10392851	0.99460428	0.10374152	1.00000000
0.000	0.520	0.08684250	1.09983959	0.09575366	1.10399997
0.040	0.520	0.08747701	1.09180921	0.09575257	1.09599997
0.080	0.520	0.08812088	1.08377838	0.09575148	1.08799996
0.120	0.520	0.08877432	1.07574708	0.09575038	1.07999998
0.160	0.520	0.08943754	1.06771531	0.09574927	1.07199998
0.200	0.520	0.09011078	1.05968306	0.09574816	1.06399998
0.240	0.520	0.09079425	1.05165034	0.09574704	1.05599997
0.280	0.520	0.09148819	1.04361709	0.09574592	1.04799997
0.320	0.520	0.09219285	1.03558334	0.09574479	1.03999997
0.360	0.520	0.09290847	1.02754905	0.09574366	1.03199997
0.400	0.520	0.09363533	1.01951424	0.09574252	1.02399997
0.440	0.520	0.09437367	1.01147887	0.09574138	1.01599997
0.480	0.520	0.09512378	1.00344292	0.09574023	1.00799997
0.520	0.520	0.09588595	0.99540645	0.09573909	1.00000000
0.000	0.560	0.07900295	1.10853151	0.08775992	1.11199997
0.040	0.560	0.07957554	1.10050639	0.08775871	1.10399997
0.080	0.560	0.08015651	1.09248089	0.08775749	1.09599997
0.120	0.560	0.08074605	1.08445505	0.08775626	1.08799998

0.160	0.560	0.08134433	1.07642881	0.08775503	1.07999998
0.200	0.560	0.08195157	1.06840218	0.08775378	1.07199998
0.240	0.560	0.08256796	1.06037513	0.08775252	1.06399998
0.280	0.560	0.08319371	1.05234768	0.08775125	1.05599997
0.320	0.560	0.08382903	1.04431980	0.08774996	1.04799997
0.360	0.560	0.08447416	1.03629150	0.08774867	1.03999997
0.400	0.560	0.08512931	1.02826276	0.08774737	1.03199997
0.440	0.560	0.08579473	1.02023357	0.08774606	1.02399997
0.480	0.560	0.08647065	1.01220391	0.08774474	1.01599997
0.520	0.560	0.08715734	1.00417381	0.08774341	1.00799998
0.560	0.560	0.08785504	0.99614322	0.08774206	1.00000000
0.000	0.600	0.07128391	1.11715558	0.07977038	1.11999997
0.040	0.600	0.07179649	1.10913516	0.07976912	1.11199997
0.080	0.600	0.07231650	1.10111443	0.07976784	1.10399997
0.120	0.600	0.07284411	1.09309342	0.07976656	1.09599998
0.160	0.600	0.07337949	1.08507210	0.07976525	1.08799998
0.200	0.600	0.07392281	1.07705044	0.07976394	1.07999998
0.240	0.600	0.07447424	1.06902846	0.07976261	1.07199998
0.280	0.600	0.07503398	1.06100615	0.07976126	1.06399998
0.320	0.600	0.07560221	1.05298352	0.07975990	1.05599997
0.360	0.600	0.07617913	1.04496052	0.07975853	1.04799997
0.400	0.600	0.07676494	1.03693719	0.07975714	1.03999997
0.440	0.600	0.07735984	1.02891348	0.07975575	1.03199997
0.480	0.600	0.07796405	1.02088940	0.07975433	1.02399997
0.520	0.600	0.07857779	1.01286496	0.07975290	1.01599998
0.560	0.600	0.07920129	1.00484012	0.07975146	1.00799998
0.600	0.600	0.07983478	0.99681489	0.07975000	1.00000000
0.000	0.640	0.06368180	1.12571351	0.07178453	1.12799997
0.040	0.640	0.06413616	1.11769722	0.07178326	1.11999997
0.080	0.640	0.06459706	1.10968069	0.07178198	1.11199997
0.120	0.640	0.06506464	1.10166394	0.07178069	1.10399997
0.160	0.640	0.06553905	1.09364695	0.07177938	1.09599998
0.200	0.640	0.06602043	1.08562970	0.07177806	1.08799998
0.240	0.640	0.06650895	1.07761219	0.07177672	1.07999998
0.280	0.640	0.06700476	1.06959442	0.07177536	1.07199998
0.320	0.640	0.06750803	1.06157638	0.07177399	1.06399998
0.360	0.640	0.06801892	1.05355808	0.07177261	1.05599997
0.400	0.640	0.06853762	1.04553948	0.07177120	1.04799997
0.440	0.640	0.06906430	1.03752061	0.07176978	1.03999997
0.480	0.640	0.06959915	1.02950145	0.07176834	1.03199997
0.520	0.640	0.07014236	1.02148200	0.07176689	1.02399998
0.560	0.640	0.07069413	1.01346223	0.07176542	1.01599998
0.600	0.640	0.07125466	1.00544214	0.07176393	1.00799998
0.640	0.640	0.07182416	0.99742174	0.07176243	1.00000000
0.000	0.680	0.05619316	1.13420689	0.06380183	1.13599997
0.040	0.680	0.05659103	1.12619420	0.06380061	1.12799997
0.080	0.680	0.05699458	1.11818136	0.06379938	1.11999997
0.120	0.680	0.05740394	1.11016832	0.06379813	1.11199997
0.160	0.680	0.05781923	1.10215511	0.06379687	1.10399998
0.200	0.680	0.05824057	1.09414170	0.06379559	1.09599998
0.240	0.680	0.05866811	1.08612810	0.06379429	1.08799998
0.280	0.680	0.05910197	1.07811428	0.06379298	1.07999998
0.320	0.680	0.05954231	1.07010026	0.06379165	1.07199998
0.360	0.680	0.05998927	1.06208603	0.06379030	1.06399998
0.400	0.680	0.06044299	1.05407159	0.06378894	1.05599997
0.440	0.680	0.06090364	1.04605692	0.06378756	1.04799997

0.480	0.680	0.06137137	1.03804202	0.06378616	1.03999997
0.520	0.680	0.06184634	1.03002692	0.06378474	1.03199999
0.560	0.680	0.06232874	1.02201157	0.06378331	1.02399998
0.600	0.680	0.06281872	1.01399597	0.06378185	1.01599998
0.640	0.680	0.06331649	1.00598011	0.06378038	1.00799998
0.680	0.680	0.06382221	0.99796404	0.06377889	1.00000000
0.000	0.720	0.04881468	1.14263723	0.05582181	1.14399997
0.040	0.720	0.04915772	1.13462768	0.05582068	1.13599997
0.080	0.720	0.04950562	1.12661798	0.05581953	1.12799997
0.120	0.720	0.04985848	1.11860816	0.05581836	1.11999997
0.160	0.720	0.05021641	1.11059820	0.05581719	1.11199998
0.200	0.720	0.05057953	1.10258810	0.05581599	1.10399998
0.240	0.720	0.05094794	1.09457783	0.05581478	1.09599998
0.280	0.720	0.05132175	1.08656743	0.05581356	1.08799998
0.320	0.720	0.05170110	1.07855686	0.05581231	1.07999998
0.360	0.720	0.05208610	1.07054615	0.05581106	1.07199998
0.400	0.720	0.05247688	1.06253527	0.05580978	1.06399998
0.440	0.720	0.05287358	1.05452422	0.05580849	1.05599997
0.480	0.720	0.05327632	1.04651302	0.05580718	1.04799997
0.520	0.720	0.05368526	1.03850165	0.05580585	1.03999999
0.560	0.720	0.05410052	1.03049008	0.05580450	1.03199999
0.600	0.720	0.05452226	1.02247834	0.05580314	1.02399998
0.640	0.720	0.05495064	1.01446641	0.05580175	1.01599998
0.680	0.720	0.05538581	1.00645430	0.05580035	1.00799998
0.720	0.720	0.05582793	0.99844202	0.05579893	1.00000000
0.000	0.760	0.04154320	1.15100602	0.04784400	1.15199998
0.040	0.760	0.04183299	1.14299911	0.04784298	1.14399997
0.080	0.760	0.04212685	1.13499210	0.04784195	1.13599997
0.120	0.760	0.04242488	1.12698498	0.04784091	1.12799997
0.160	0.760	0.04272715	1.11897780	0.04783985	1.11999998
0.200	0.760	0.04303376	1.11097046	0.04783877	1.11199998
0.240	0.760	0.04334481	1.10296306	0.04783769	1.10399998
0.280	0.760	0.04366039	1.09495553	0.04783658	1.09599998
0.320	0.760	0.04398060	1.08694788	0.04783547	1.08799998
0.360	0.760	0.04430554	1.07894013	0.04783433	1.07999998
0.400	0.760	0.04463533	1.07093226	0.04783318	1.07199998
0.440	0.760	0.04497006	1.06292428	0.04783202	1.06399998
0.480	0.760	0.04530986	1.05491618	0.04783084	1.05599997
0.520	0.760	0.04565483	1.04690796	0.04782964	1.04799999
0.560	0.760	0.04600510	1.03889961	0.04782843	1.03999999
0.600	0.760	0.04636079	1.03089113	0.04782719	1.03199999
0.640	0.760	0.04672202	1.02288250	0.04782594	1.02399998
0.680	0.760	0.04708893	1.01487375	0.04782468	1.01599998
0.720	0.760	0.04746166	1.00686489	0.04782339	1.00800000
0.760	0.760	0.04784033	0.99885586	0.04782208	1.00000000
0.000	0.800	0.03437570	1.15931466	0.03986796	1.15999998
0.040	0.800	0.03461375	1.15130992	0.03986708	1.15199998
0.080	0.800	0.03485513	1.14330512	0.03986619	1.14399997
0.120	0.800	0.03509990	1.13530026	0.03986529	1.13599997
0.160	0.800	0.03534812	1.12729533	0.03986438	1.12799999
0.200	0.800	0.03559989	1.11929033	0.03986345	1.11999998

0.240	0.800	0.03585527	1.11128526	0.03986252	1.11199998
0.280	0.800	0.03611434	1.10328011	0.03986156	1.10399998
0.320	0.800	0.03637718	1.09527488	0.03986060	1.09599998
0.360	0.800	0.03664388	1.08726958	0.03985962	1.08799998
0.400	0.800	0.03691452	1.07926420	0.03985863	1.07999998
0.440	0.800	0.03718919	1.07125875	0.03985763	1.07199998
0.480	0.800	0.03746798	1.06325320	0.03985660	1.06399998
0.520	0.800	0.03775098	1.05524760	0.03985557	1.05599999
0.560	0.800	0.03803829	1.04724189	0.03985452	1.04799999
0.600	0.800	0.03833001	1.03923609	0.03985345	1.03999999
0.640	0.800	0.03862625	1.03123021	0.03985237	1.03199999
0.680	0.800	0.03892709	1.02322423	0.03985128	1.02399998
0.720	0.800	0.03923267	1.01521818	0.03985016	1.01600000
0.760	0.800	0.03954307	1.00721201	0.03984903	1.00800000
0.800	0.800	0.03985844	0.99920575	0.03984788	1.00000000
0.000	0.840	0.02730927	1.16756445	0.03189326	1.16799998
0.040	0.840	0.02749704	1.15956147	0.03189254	1.15999998
0.080	0.840	0.02768740	1.15155844	0.03189181	1.15199998
0.120	0.840	0.02788043	1.14355537	0.03189108	1.14399997
0.160	0.840	0.02807616	1.13555227	0.03189033	1.13599999
0.200	0.840	0.02827467	1.12754911	0.03188957	1.12799999
0.240	0.840	0.02847600	1.11954592	0.03188881	1.11999998
0.280	0.840	0.02868022	1.11154267	0.03188803	1.11199998
0.320	0.840	0.02888739	1.10353937	0.03188724	1.10399998
0.360	0.840	0.02909757	1.09553603	0.03188644	1.09599998
0.400	0.840	0.02931084	1.08753263	0.03188563	1.08799998
0.440	0.840	0.02952726	1.07952919	0.03188480	1.07999998
0.480	0.840	0.02974689	1.07152570	0.03188397	1.07199998
0.520	0.840	0.02996983	1.06352217	0.03188312	1.06399999
0.560	0.840	0.03019612	1.05551858	0.03188226	1.05599999
0.600	0.840	0.03042587	1.04751493	0.03188139	1.04799999
0.640	0.840	0.03065913	1.03951121	0.03188050	1.03999999
0.680	0.840	0.03089600	1.03150746	0.03187960	1.03199999
0.720	0.840	0.03113656	1.02350366	0.03187869	1.02400000
0.760	0.840	0.03138090	1.01549977	0.03187776	1.01600000
0.800	0.840	0.03162910	1.00749582	0.03187682	1.00800000
0.840	0.840	0.03188127	0.99949183	0.03187587	1.00000000
0.000	0.880	0.02034111	1.17575669	0.02391949	1.17599998
0.040	0.880	0.02047998	1.16775503	0.02391895	1.16799998
0.080	0.880	0.02062077	1.15975335	0.02391840	1.15999998
0.120	0.880	0.02076351	1.15175165	0.02391784	1.15199998
0.160	0.880	0.02090823	1.14374993	0.02391727	1.14399999
0.200	0.880	0.02105499	1.13574819	0.02391670	1.13599999
0.240	0.880	0.02120382	1.12774640	0.02391612	1.12799999
0.280	0.880	0.02135477	1.11974461	0.02391552	1.11999998
0.320	0.880	0.02150789	1.11174279	0.02391493	1.11199998
0.360	0.880	0.02166321	1.10374093	0.02391432	1.10399998
0.400	0.880	0.02182080	1.09573905	0.02391370	1.09599998
0.440	0.880	0.02198070	1.08773714	0.02391308	1.08799998
0.480	0.880	0.02214296	1.07973521	0.02391244	1.07999998
0.520	0.880	0.02230763	1.07173326	0.02391180	1.07199999
0.560	0.880	0.02247477	1.06373126	0.02391115	1.06399999
0.600	0.880	0.02264444	1.05572925	0.02391048	1.05599999
0.640	0.880	0.02281668	1.04772719	0.02390981	1.04799999
0.680	0.880	0.02299157	1.03972512	0.02390913	1.03999999
0.720	0.880	0.02316916	1.03172300	0.02390843	1.03200000

0.760	0.880	0.02334951	1.02372086	0.02390773	1.02400000
0.800	0.880	0.02353270	1.01571868	0.02390701	1.01600000
0.840	0.880	0.02371878	1.00771646	0.02390629	1.00800000
0.880	0.880	0.02390783	0.99971421	0.02390555	1.00000000
0.000	0.920	0.01346854	1.18389257	0.01594627	1.18399998
0.040	0.920	0.01355986	1.17589186	0.01594591	1.17599998
0.080	0.920	0.01365243	1.16789111	0.01594554	1.16799998
0.120	0.920	0.01374627	1.15989038	0.01594517	1.15999998
0.160	0.920	0.01384141	1.15188963	0.01594479	1.15199999
0.200	0.920	0.01393787	1.14388886	0.01594441	1.14399999
0.240	0.920	0.01403569	1.13588808	0.01594402	1.13599999
0.280	0.920	0.01413489	1.12788729	0.01594362	1.12799999
0.320	0.920	0.01423550	1.11988650	0.01594322	1.11999998
0.360	0.920	0.01433756	1.11188568	0.01594282	1.11199998
0.400	0.920	0.01444109	1.10388486	0.01594241	1.10399998
0.440	0.920	0.01454612	1.09588402	0.01594199	1.09599998
0.480	0.920	0.01465270	1.08788317	0.01594156	1.08799998
0.520	0.920	0.01476085	1.07988233	0.01594113	1.07999999
0.560	0.920	0.01487060	1.07188145	0.01594070	1.07199999
0.600	0.920	0.01498200	1.06388057	0.01594025	1.06399999
0.640	0.920	0.01509509	1.05587966	0.01593980	1.05599999
0.680	0.920	0.01520989	1.04787875	0.01593935	1.04799999
0.720	0.920	0.01532645	1.03987784	0.01593889	1.04000000
0.760	0.920	0.01544481	1.03187690	0.01593841	1.03200000
0.800	0.920	0.01556502	1.02387595	0.01593794	1.02400000
0.840	0.920	0.01568711	1.01587498	0.01593745	1.01600000
0.880	0.920	0.01581113	1.00787399	0.01593696	1.00800000
0.920	0.920	0.01593713	0.99987300	0.01593646	1.00000000
0.000	0.960	0.00668900	1.19197331	0.00797323	1.19199998
0.040	0.960	0.00673404	1.18397311	0.00797305	1.18399998
0.080	0.960	0.00677970	1.17597295	0.00797287	1.17599998
0.120	0.960	0.00682598	1.16797275	0.00797268	1.16799998
0.160	0.960	0.00687289	1.15997259	0.00797249	1.15999999
0.200	0.960	0.00692046	1.15197239	0.00797230	1.15199999
0.240	0.960	0.00696868	1.14397220	0.00797211	1.14399999
0.280	0.960	0.00701759	1.13597200	0.00797191	1.13599999
0.320	0.960	0.00706718	1.12797181	0.00797171	1.12799999
0.360	0.960	0.00711748	1.11997161	0.00797151	1.11999998
0.400	0.960	0.00716850	1.11197140	0.00797131	1.11199998
0.440	0.960	0.00722026	1.10397119	0.00797110	1.10399998
0.480	0.960	0.00727277	1.09597098	0.00797089	1.09599998
0.520	0.960	0.00732606	1.08797079	0.00797068	1.08799999
0.560	0.960	0.00738012	1.07997056	0.00797046	1.07999999
0.600	0.960	0.00743500	1.07197035	0.00797024	1.07199999
0.640	0.960	0.00749069	1.06397013	0.00797002	1.06399999
0.680	0.960	0.00754722	1.05596990	0.00796979	1.05599999
0.720	0.960	0.00760462	1.04796969	0.00796956	1.04800000
0.760	0.960	0.00766289	1.03996945	0.00796933	1.04000000
0.800	0.960	0.00772206	1.03196921	0.00796909	1.03200000
0.840	0.960	0.00778216	1.02396897	0.00796885	1.02400000
0.880	0.960	0.00784320	1.01596873	0.00796861	1.01600000
0.920	0.960	0.00790520	1.00796851	0.00796836	1.00800001
0.960	0.960	0.00796819	0.99996825	0.00796811	1.00000000
-0.000	1.000	0.00000000	1.19999998	0.00000000	1.19999998
0.040	1.000	0.00000000	1.19199998	0.00000000	1.19199998
0.080	1.000	0.00000000	1.18399998	0.00000000	1.18399998

0.120	1.000	0.00000000	1.17599998	0.00000000	1.17599998
0.160	1.000	0.00000000	1.16799999	0.00000000	1.16799999
0.200	1.000	0.00000000	1.15999999	0.00000000	1.15999999
0.240	1.000	0.00000000	1.15199999	0.00000000	1.15199999
0.280	1.000	0.00000000	1.14399999	0.00000000	1.14399999
0.320	1.000	0.00000000	1.13599999	0.00000000	1.13599999
0.360	1.000	0.00000000	1.12799999	0.00000000	1.12799999
0.400	1.000	0.00000000	1.11999998	0.00000000	1.11999998
0.440	1.000	0.00000000	1.11199998	0.00000000	1.11199998
0.480	1.000	0.00000000	1.10399998	0.00000000	1.10399998
0.520	1.000	0.00000000	1.09600000	0.00000000	1.09600000
0.560	1.000	0.00000000	1.08799999	0.00000000	1.08799999
0.600	1.000	0.00000000	1.07999999	0.00000000	1.07999999
0.640	1.000	0.00000000	1.07199999	0.00000000	1.07199999
0.680	1.000	0.00000000	1.06399999	0.00000000	1.06399999
0.720	1.000	0.00000000	1.05600000	0.00000000	1.05600000
0.760	1.000	0.00000000	1.04800000	0.00000000	1.04800000
0.800	1.000	0.00000000	1.04000000	0.00000000	1.04000000
0.840	1.000	0.00000000	1.03200000	0.00000000	1.03200000
0.880	1.000	0.00000000	1.02400000	0.00000000	1.02400000
0.920	1.000	0.00000000	1.01600001	0.00000000	1.01600001
0.960	1.000	0.00000000	1.00800001	0.00000000	1.00800001

COMPUTER PROGRAM B

```
      READ (5,100) E,H,N
      HH=H*H
      G=0.96
      WRITE (6,101)
      CC=(1.-G)*16./3.14159265/3.14159265
      CB=2.*CC/3.14159265
      TQ=0.
      J=N
      Y=1.+H/2.
   10 J=J-1
      Y=Y-H
      X=-H/2.
      DO 30 I=1,J
      X=X+H
      IF (I.LT.J) GO TO 12
      YY=Y
      Y=Y+H/6.
      X=X-H/6.
      HH=0.5*HH
   12 SG=-1.0
      WW=0.
      WX=0.
      WY=0.
      DO 20 NI=1,8
      FN=NI-1
      GG=(FN+0.5)*3.14159265
      ANC=0.5*(EXP(GG)+EXP(-GG))*(1.+2.*FN)*(1.+2.*FN)
      ANB=ANC*(1.+2.*FN)
      GGX=GG*X
      GGY=GG*Y
   15 SHY=0.5*(EXP(GGY)-EXP(-GGY))
      CHY=EXP(-GGY)+SHY
      SG=-1.*SG
      WW=WW+SG*SHY*SIN(GGX)/ANB
      WX=WX+SG*SHY*COS(GGX)/ANC
   20 WY=WY+SG*CHY*SIN(GGX)/ANC
      EX=-2.*G*(Y-X)-CC*WX
      EY=2.*X*(1.-G)-CC*WY
      W=-G*X*(Y-X)+(1.-G)*X*Y-CB*WW
      S=EX*EX+EY*EY
      SS=SQRT(S)
      B=E+1./SS
      SX=B*EX
      SY=B*EY
      ST=SQRT(SX*SX+SY*SY)
      TQ=TQ+(SY*X-SX*Y)*HH
   30 WRITE (6,102) X,Y,W,SX,SY,ST
      WRITE (6,103)
   32 Y=YY
      HH=2.*HH
      IF (J.GT.1) GO TO 10
```

```
      TQQ=TQ*8.
      WRITE (6,104) TQQ
      WRITE (6,105) TQ
      STOP
  100 FORMAT (F10.3,F9.3,I6)
  101 FORMAT (74H                   X         Y              W                SX
     1     SY             ST/)
  102 FORMAT (8X,2F8.3,2F14.8,2F13.8)
  103 FORMAT (1HJ)
  104 FORMAT (31H            DIMENSIONLESS TORQUE =,F12.8)
  105 FORMAT (31H            ONE-EIGHTH OF IT     =,F12.8)
      END
```

DATA CARD -

```
    0.100    0.020    51
```

Table A.2

RESULTS FROM PROGRAM B

x	y	w	s_{xz}	s_{yz}	τ
0.030	0.990	-0.02805391	-1.18963275	0.00005464	1.18963275
0.070	0.990	-0.06277041	-1.18194842	0.00009264	1.18194842
0.110	0.990	-0.09441099	-1.17425154	0.00007282	1.17425154
0.150	0.990	-0.12296902	-1.16653510	0.00002812	1.16653510
0.190	0.990	-0.14843657	-1.15879999	0.00002084	1.15879999
0.230	0.990	-0.17080761	-1.15105420	0.00008623	1.15105420
0.270	0.990	-0.19007957	-1.14330528	0.00019541	1.14330528
0.310	0.990	-0.20625146	-1.13555298	0.00027515	1.13555301
0.350	0.990	-0.21932039	-1.12778860	0.00027356	1.12778863
0.390	0.990	-0.22927941	-1.12000223	0.00021604	1.12000225
0.430	0.990	-0.23611893	-1.11219161	0.00019626	1.11219163
0.470	0.990	-0.23983050	-1.10436429	0.00029875	1.10436432
0.510	0.990	-0.24040958	-1.09653037	0.00051326	1.09653049
0.550	0.990	-0.23785440	-1.08869212	0.00072228	1.08869236
0.590	0.990	-0.23216159	-1.08083954	0.00079370	1.08083982
0.630	0.990	-0.22332221	-1.07295688	0.00072014	1.07295711
0.670	0.990	-0.21132207	-1.06503556	0.00068638	1.06503577
0.710	0.990	-0.19614656	-1.05708213	0.00097537	1.05708257
0.750	0.990	-0.17778601	-1.04911182	0.00173759	1.04911325
0.790	0.990	-0.15623607	-1.04113145	0.00277893	1.04113516
0.830	0.990	-0.13149073	-1.03312508	0.00358937	1.03313131
0.870	0.990	-0.10353220	-1.02505423	0.00382050	1.02506133
0.910	0.990	-0.07232507	-1.01687207	0.00463588	1.01688264
0.950	0.990	-0.03782147	-1.00846590	0.01523320	1.00858093
0.987	0.993	-0.00653873	-0.98996323	0.15190378	1.00154976
0.030	0.950	-0.02685800	-1.18164528	0.00008094	1.18164528
0.070	0.950	-0.05997844	-1.17395524	0.00018129	1.17395524
0.110	0.950	-0.09002070	-1.16625380	0.00026745	1.16625382
0.150	0.950	-0.11697963	-1.15853834	0.00035366	1.15853838
0.190	0.950	-0.14084956	-1.15080904	0.00046569	1.15080913
0.230	0.950	-0.16162549	-1.14306874	0.00061921	1.14306890
0.270	0.950	-0.17930357	-1.13531991	0.00080583	1.13532020
0.310	0.950	-0.19388046	-1.12756213	0.00100123	1.12756258
0.350	0.950	-0.20535195	-1.11979183	0.00119090	1.11979246
0.390	0.950	-0.21371220	-1.11200509	0.00139206	1.11200596
0.430	0.950	-0.21895431	-1.10420073	0.00165007	1.10420195
0.470	0.950	-0.22107167	-1.09638099	0.00200862	1.09638282
0.510	0.950	-0.22005887	-1.08854882	0.00247802	1.08855164
0.550	0.950	-0.21591121	-1.08070397	0.00303395	1.08070822
0.590	0.950	-0.20862295	-1.07284153	0.00365969	1.07284776
0.630	0.950	-0.19818592	-1.06495444	0.00440892	1.06496356
0.670	0.950	-0.18458968	-1.05703781	0.00544428	1.05705183
0.710	0.950	-0.16782330	-1.04909057	0.00702289	1.04911407
0.750	0.950	-0.14787714	-1.04111129	0.00945957	1.04115426
0.790	0.950	-0.12474235	-1.03308749	0.01319230	1.03317171
0.830	0.950	-0.09840790	-1.02497515	0.01925690	1.02515602
0.870	0.950	-0.06885661	-1.01660764	0.03130603	1.01708954
0.910	0.950	-0.03606374	-1.00665608	0.06817853	1.00896221
0.947	0.953	-0.00626325	-0.95153497	0.31455589	1.00217972
0.030	0.910	-0.02566421	-1.17366558	0.00012729	1.17366558
0.070	0.910	-0.05719239	-1.16597355	0.00030372	1.16597358
0.110	0.910	-0.08564164	-1.15827062	0.00048792	1.15827071
0.150	0.910	-0.11100738	-1.15055574	0.00068819	1.15055593

0.190	0.910	-0.13328481	-1.14282889	0.00091785	1.14282925
0.230	0.910	-0.15246930	-1.13509090	0.00118715	1.13509151
0.270	0.910	-0.16855661	-1.12734246	0.00149827	1.12734344
0.310	0.910	-0.18154256	-1.11958307	0.00184891	1.11958459
0.350	0.910	-0.19142253	-1.11181111	0.00224287	1.11181336
0.390	0.910	-0.19819119	-1.10402470	0.00269932	1.10402800
0.430	0.910	-0.20184270	-1.09622292	0.00325243	1.09622773
0.470	0.910	-0.20237114	-1.08840593	0.00394171	1.08841305
0.510	0.910	-0.19977086	-1.08057382	0.00480295	1.08058449
0.550	0.910	-0.19403616	-1.07272513	0.00587385	1.07274121
0.590	0.910	-0.18516067	-1.06485620	0.00722106	1.06488068
0.630	0.910	-0.17313677	-1.05696146	0.00898366	1.05699963
0.670	0.910	-0.15795569	-1.04903373	0.01142522	1.04909594
0.710	0.910	-0.13960808	-1.04106107	0.01500901	1.04116925
0.750	0.910	-0.11808454	-1.03301465	0.02058808	1.03321978
0.790	0.910	-0.09337520	-1.02480581	0.03004452	1.02524612
0.830	0.910	-0.06546848	-1.01607021	0.04887267	1.01724490
0.870	0.910	-0.03434979	-1.00402070	0.10232536	1.00922149
0.907	0.913	-0.00597513	-0.92261163	0.39246111	1.00261555
0.030	0.870	-0.02447318	-1.16569536	0.00018377	1.16569536
0.070	0.870	-0.05441318	-1.15800276	0.00044554	1.15800283
0.110	0.870	-0.08127405	-1.15029950	0.00072981	1.15029972
0.150	0.870	-0.10505149	-1.14258517	0.00104412	1.14258565
0.190	0.870	-0.12574105	-1.13485959	0.00139896	1.13486045
0.230	0.870	-0.14333832	-1.12712286	0.00180501	1.12712430
0.270	0.870	-0.15783894	-1.11937503	0.00227178	1.11937732
0.310	0.870	-0.16923850	-1.11161552	0.00280968	1.11161905
0.350	0.870	-0.17753231	-1.10384337	0.00343519	1.10384871
0.390	0.870	-0.18271535	-1.09605738	0.00417598	1.09606532
0.430	0.870	-0.18478221	-1.08825644	0.00507316	1.08826826
0.470	0.870	-0.18372735	-1.08043965	0.00618146	1.08045732
0.510	0.870	-0.17954514	-1.07260550	0.00757235	1.07263223
0.550	0.870	-0.17222968	-1.06475104	0.00934823	1.06479208
0.590	0.870	-0.16177458	-1.05687089	0.01167516	1.05693536
0.630	0.870	-0.14817272	-1.04895535	0.01484419	1.04906037
0.670	0.870	-0.13141624	-1.04098519	0.01938896	1.04116573
0.710	0.870	-0.11149676	-1.03291487	0.02635706	1.03325110
0.750	0.870	-0.08840546	-1.02460853	0.03810139	1.02531670
0.790	0.870	-0.06213288	-1.01551657	0.06129912	1.01736497
0.830	0.870	-0.03266849	-1.00160877	0.12527246	1.00941236
0.867	0.873	-0.00569148	-0.90373711	0.43488313	1.00292776
0.030	0.830	-0.02328513	-1.15773518	0.00024682	1.15773519
0.070	0.830	-0.05164104	-1.15004250	0.00060194	1.15004265
0.110	0.830	-0.07691788	-1.14233939	0.00099277	1.14233982
0.150	0.830	-0.09911148	-1.13462562	0.00142844	1.13462652
0.190	0.830	-0.11821761	-1.12690094	0.00192079	1.12690258
0.230	0.830	-0.13423198	-1.11916518	0.00248386	1.11916793
0.270	0.830	-0.14715028	-1.11141803	0.00313421	1.11142244
0.310	0.830	-0.15696810	-1.10365892	0.00389289	1.10366578
0.350	0.830	-0.16368084	-1.09588691	0.00478920	1.09589737
0.390	0.830	-0.16728364	-1.08810085	0.00586544	1.08811664
0.430	0.830	-0.16777142	-1.08029919	0.00718226	1.08032307
0.470	0.830	-0.16513889	-1.07247990	0.00882605	1.07251621
0.510	0.830	-0.15938058	-1.06463958	0.01092342	1.06469561
0.550	0.830	-0.15049070	-1.05677215	0.01367107	1.05686058
0.590	0.830	-0.13846312	-1.04886591	0.01739763	1.04901018
0.630	0.830	-0.12329118	-1.04089620	0.02269515	1.04114358
0.670	0.830	-0.10496773	-1.03280331	0.03073202	1.03326043

0.710	0.830	-0.08348517	-1.02441082	0.04414581	1.02536158
0.750	0.830	-0.05883547	-1.01501539	0.07036674	1.01745158
0.790	0.830	-0.03101011	-0.99961906	0.14128029	1.00955355
0.827	0.833	-0.00541296	-0.89151434	0.45991788	1.00315615
0.030	0.790	-0.02210008	-1.14978502	0.00031539	1.14978505
0.070	0.790	-0.04887597	-1.14209249	0.00077194	1.14209274
0.110	0.790	-0.07257293	-1.13438974	0.00127834	1.13439045
0.150	0.790	-0.09318694	-1.12667655	0.00184670	1.12667806
0.190	0.790	-0.11071391	-1.11895267	0.00249248	1.11895543
0.230	0.790	-0.12514969	-1.11121773	0.00323520	1.11122244
0.270	0.790	-0.13649008	-1.10347123	0.00409980	1.10347883
0.310	0.790	-0.14473076	-1.09571242	0.00511937	1.09572437
0.350	0.790	-0.14986725	-1.08794018	0.00633954	1.08795864
0.390	0.790	-0.15189492	-1.08015286	0.00782512	1.08018121
0.430	0.790	-0.15080891	-1.07234805	0.00967016	1.07239165
0.470	0.790	-0.14660419	-1.06452177	0.01201490	1.06458957
0.510	0.790	-0.13927551	-1.05666701	0.01507750	1.05677457
0.550	0.790	-0.12881736	-1.04877015	0.01921797	1.04894621
0.590	0.790	-0.11522394	-1.04080215	0.02507687	1.04110419
0.630	0.790	-0.09848909	-1.03269198	0.03391057	1.03324858
0.670	0.790	-0.07860628	-1.02423089	0.04854749	1.02538080
0.710	0.790	-0.05556865	-1.01459540	0.07691380	1.01750653
0.750	0.790	-0.02936903	-0.99806504	0.15253042	1.00965309
0.787	0.793	-0.00513803	-0.88347022	0.47553545	1.00332127
0.030	0.750	-0.02091798	-1.14184471	0.00038944	1.14184477
0.070	0.750	-0.04611782	-1.13415244	0.00095625	1.13415284
0.110	0.750	-0.06823893	-1.12645013	0.00158927	1.12645125
0.150	0.750	-0.08727743	-1.11873757	0.00230479	1.11873994
0.190	0.750	-0.10322938	-1.11101439	0.00312372	1.11101877
0.230	0.750	-0.11609079	-1.10328011	0.00407328	1.10328762
0.270	0.750	-0.12585757	-1.09553398	0.00518946	1.09554627
0.310	0.750	-0.13252556	-1.08777494	0.00652124	1.08779448
0.350	0.750	-0.13609046	-1.08000132	0.00813729	1.08003196
0.390	0.750	-0.13654784	-1.07221052	0.01013690	1.07225842
0.430	0.750	-0.13389308	-1.06439813	0.01266859	1.06447352
0.470	0.750	-0.12812140	-1.05655643	0.01596397	1.05667702
0.510	0.750	-0.11922785	-1.04867021	0.02040457	1.04886870
0.550	0.750	-0.10720723	-1.04070703	0.02666610	1.04104860
0.590	0.750	-0.09205414	-1.03258749	0.03606773	1.03321720
0.630	0.750	-0.07376290	-1.02407886	0.05156840	1.02537642
0.670	0.750	-0.05232759	-1.01426979	0.08142275	1.01753273
0.710	0.750	-0.02774206	-0.99692785	0.16019258	1.00971619
0.747	0.753	-0.00486574	-0.87820502	0.48542501	1.00343485
0.030	0.710	-0.01973875	-1.13391399	0.00046941	1.13391408
0.070	0.710	-0.04336638	-1.12622198	0.00115653	1.12622256
0.110	0.710	-0.06391554	-1.11852015	0.00192944	1.11852180
0.150	0.710	-0.08138247	-1.11080817	0.00280998	1.11081172
0.190	0.710	-0.09576339	-1.10308559	0.00382661	1.10309222
0.230	0.710	-0.10705448	-1.09535166	0.00501711	1.09536315
0.270	0.710	-0.11525182	-1.08760534	0.00643280	1.08762437
0.310	0.710	-0.12035142	-1.07984496	0.00814525	1.07987567
0.350	0.710	-0.12234917	-1.07206779	0.01025751	1.07211685
0.390	0.710	-0.12124086	-1.06426930	0.01292339	1.06434775
0.430	0.710	-0.11702213	-1.05644117	0.01638268	1.05656819
0.470	0.710	-0.10968846	-1.04856725	0.02103013	1.04877811
0.510	0.710	-0.09923521	-1.04061274	0.02756376	1.04097773
0.550	0.710	-0.08565754	-1.03249271	0.03734312	1.03316779
0.590	0.710	-0.06895044	-1.02395859	0.05341008	1.02535058

0.630	0.710	-0.04910874	-1.01404149	0.08422368	1.01753318
0.670	0.710	-0.02612708	-0.99617953	0.16497271	1.00974732
0.707	0.713	-0.00459561	-0.87495660	0.49139864	1.00350469
0.030	0.670	-0.01856229	-1.12599249	0.00055609	1.12599262
0.070	0.670	-0.04062143	-1.11830078	0.00137525	1.11830161
0.110	0.670	-0.05960235	-1.11059939	0.00230409	1.11060178
0.150	0.670	-0.07550149	-1.10288792	0.00337183	1.10289306
0.190	0.670	-0.08831523	-1.09516564	0.00461720	1.09517537
0.230	0.670	-0.09803991	-1.08743156	0.00609284	1.08744862
0.270	0.670	-0.10467180	-1.07968400	0.00787201	1.07971268
0.310	0.670	-0.10820711	-1.07192024	0.01005998	1.07196745
0.350	0.670	-0.10864193	-1.06413565	0.01281355	1.06421279
0.390	0.670	-0.10597229	-1.05632174	0.01637686	1.05644866
0.430	0.670	-0.10019408	-1.04846182	0.02115151	1.04867514
0.470	0.670	-0.09130309	-1.04051999	0.02784703	1.04089255
0.510	0.670	-0.07929496	-1.03240841	0.03784387	1.03310176
0.550	0.670	-0.06416524	-1.02387031	0.05422660	1.02530530
0.590	0.670	-0.04590933	-1.01390704	0.08555839	1.01751054
0.630	0.670	-0.02452251	-0.99578736	0.16734055	1.00975012
0.667	0.673	-0.00432732	-0.87328900	0.49442115	1.00353671
0.030	0.630	-0.01738849	-1.11807985	0.00065057	1.11808003
0.070	0.630	-0.03788267	-1.11038841	0.00161579	1.11038957
0.110	0.630	-0.05529895	-1.10268740	0.00272028	1.10269075
0.150	0.630	-0.06963391	-1.09497620	0.00400320	1.09498351
0.190	0.630	-0.08088413	-1.08725382	0.00551762	1.08726781
0.230	0.630	-0.08904612	-1.07951864	0.00733741	1.07954357
0.270	0.630	-0.09411636	-1.07176807	0.00956857	1.07181078
0.310	0.630	-0.09609125	-1.06399744	0.01236880	1.06406933
0.350	0.630	-0.09496714	-1.05619832	0.01598339	1.05631925
0.390	0.630	-0.09074027	-1.04835411	0.02081558	1.04856073
0.430	0.630	-0.08340680	-1.04042872	0.02757732	1.04079413
0.470	0.630	-0.07296282	-1.03233402	0.03765329	1.03302046
0.510	0.630	-0.05940429	-1.02381202	0.05413610	1.02524229
0.550	0.630	-0.04272712	-1.01385936	0.08560752	1.01746717
0.590	0.630	-0.02292710	-0.99571710	0.16762368	1.00972780
0.627	0.633	-0.00406066	-0.87293921	0.49503621	1.00353561
0.030	0.590	-0.01621721	-1.11017562	0.00075433	1.11017587
0.070	0.590	-0.03514983	-1.10248440	0.00188271	1.10248599
0.110	0.590	-0.05100485	-1.09478363	0.00318761	1.09478826
0.150	0.590	-0.06377907	-1.08707238	0.00472189	1.08708263
0.190	0.590	-0.07346923	-1.07934913	0.00655906	1.07936905
0.230	0.590	-0.08007207	-1.07161138	0.00880452	1.07164755
0.270	0.590	-0.08358425	-1.06385472	0.01161498	1.06391811
0.310	0.590	-0.08400240	-1.05607095	0.01523403	1.05618081
0.350	0.590	-0.08132309	-1.04824389	0.02006193	1.04843585
0.390	0.590	-0.07554282	-1.04033838	0.02680528	1.04068365
0.430	0.590	-0.06665803	-1.03226816	0.03683829	1.03292527
0.470	0.590	-0.05466507	-1.02378033	0.05323100	1.02516324
0.510	0.590	-0.03956026	-1.01388932	0.08450826	1.01740513
0.550	0.590	-0.02133983	-0.99593463	0.16605439	1.00968304
0.587	0.593	-0.00379545	-0.87374780	0.49354573	1.00350515
0.030	0.550	-0.01504833	-1.10227935	0.00086936	1.10227969
0.070	0.550	-0.03242258	-1.09458829	0.00218222	1.09459045
0.110	0.550	-0.04671957	-1.08688750	0.00371933	1.08689387
0.150	0.550	-0.05793628	-1.07917562	0.00555300	1.07918989
0.190	0.550	-0.06606967	-1.07145033	0.00778686	1.07147862
0.230	0.550	-0.07111666	-1.06370748	0.01057490	1.06376004
0.270	0.550	-0.07307415	-1.05593934	0.01415644	1.05603422

0.310	0.550	-0.07193899	-1.04813064	0.01892465	1.04830147
0.350	0.550	-0.06770798	-1.04024790	0.02557370	1.04056221
0.390	0.550	-0.06037789	-1.03220875	0.03545415	1.03281745
0.430	0.550	-0.04994540	-1.02377092	0.05158567	1.02506974
0.470	0.550	-0.03640718	-1.01398646	0.08236700	1.01732632
0.510	0.550	-0.01975984	-0.99640640	0.16279829	1.00961823
0.547	0.553	-0.00353152	-0.87562488	0.49009199	1.00344859
0.030	0.510	-0.01388170	-1.09439055	0.00099836	1.09439100
0.070	0.510	-0.02970058	-1.08669951	0.00252289	1.08670243
0.110	0.510	-0.04244255	-1.07899837	0.00433418	1.07900707
0.150	0.510	-0.05210480	-1.07128499	0.00653302	1.07130491
0.190	0.510	-0.05868449	-1.06355564	0.00926916	1.06359602
0.230	0.510	-0.06217875	-1.05580326	0.01277527	1.05588054
0.270	0.510	-0.06258470	-1.04801371	0.01743366	1.04815870
0.310	0.510	-0.05989943	-1.04015599	0.02391958	1.04043097
0.350	0.510	-0.05411997	-1.03215332	0.03354777	1.03269836
0.390	0.510	-0.04524333	-1.02377875	0.04926170	1.02496324
0.430	0.510	-0.03326650	-1.01413926	0.07926841	1.01723247
0.470	0.510	-0.01818640	-0.99709898	0.15797365	1.00953555
0.507	0.513	-0.00326874	-0.87853109	0.48469768	1.00336869
0.030	0.470	-0.01271716	-1.08650875	0.00114504	1.08650934
0.070	0.470	-0.02698349	-1.07881753	0.00291680	1.07882146
0.110	0.470	-0.03817326	-1.07111540	0.00505936	1.07112734
0.150	0.470	-0.04628388	-1.06339907	0.00771679	1.06342707
0.190	0.470	-0.05131272	-1.05566224	0.01111295	1.05572073
0.230	0.470	-0.05325714	-1.04789225	0.01561581	1.04800859
0.270	0.470	-0.05211448	-1.04006110	0.02187566	1.04029113
0.310	0.470	-0.04788204	-1.03209897	0.03115997	1.03256922
0.350	0.470	-0.04055712	-1.02379825	0.04631142	1.02484515
0.390	0.470	-0.03013697	-1.01433548	0.07528167	1.01712526
0.430	0.470	-0.01661885	-0.99797838	0.15166466	1.00943697
0.467	0.473	-0.00300699	-0.88246611	0.47728410	1.00326792
0.030	0.430	-0.01155458	-1.07863338	0.00131465	1.07863417
0.070	0.430	-0.02427094	-1.07094168	0.00338152	1.07094702
0.110	0.430	-0.03391113	-1.06323763	0.00593577	1.06325420
0.150	0.430	-0.04047274	-1.05551590	0.00919043	1.05555590
0.190	0.430	-0.04395338	-1.04776537	0.01349585	1.04785227
0.230	0.430	-0.04435061	-1.03996156	0.01947167	1.04014383
0.270	0.430	-0.04166202	-1.03204266	0.02832723	1.03243134
0.310	0.430	-0.03588513	-1.02382348	0.04278065	1.02471688
0.350	0.430	-0.02701748	-1.01456214	0.07046502	1.01700621
0.390	0.430	-0.01505660	-0.99900934	0.14393087	1.00932440
0.427	0.433	-0.00274618	-0.88746141	0.46767433	1.00314855
0.030	0.390	-0.01039378	-1.07076390	0.00151482	1.07076495
0.070	0.390	-0.02156257	-1.06307123	0.00394347	1.06307853
0.110	0.390	-0.02965557	-1.05536372	0.00702769	1.05538712
0.150	0.390	-0.03467058	-1.04763212	0.01109709	1.04769089
0.190	0.390	-0.03660543	-1.03985558	0.01673530	1.03999024
0.230	0.390	-0.03545792	-1.03198117	0.02508305	1.03228594
0.270	0.390	-0.03122584	-1.02384822	0.03871073	1.02457976
0.310	0.390	-0.02390698	-1.01480565	0.06486918	1.01687684
0.350	0.390	-0.01349910	-1.00015446	0.13481420	1.00919958
0.387	0.393	-0.00248621	-0.89357514	0.45558540	1.00301274
0.030	0.350	-0.00923461	-1.06289973	0.00175700	1.06290118
0.070	0.350	-0.01885801	-1.05520531	0.00464419	1.05521552
0.110	0.350	-0.02540598	-1.04749153	0.00844204	1.04752553

0.150	0.350	-0.02887658	-1.03974141	0.01369298	1.03983156
0.190	0.350	-0.02926784	-1.03191114	0.02145908	1.03213424
0.230	0.350	-0.02657779	-1.02386610	0.03414014	1.02443511
0.270	0.350	-0.02080445	-1.01505197	0.05854004	1.01673862
0.310	0.350	-0.01194584	-1.00137364	0.12434439	1.00906427
0.347	0.353	-0.00222699	-0.90088760	0.44060739	1.00286257
0.030	0.310	-0.00807691	-1.05504028	0.00205924	1.05504228
0.070	0.310	-0.01615686	-1.04734273	0.00555283	1.04735745
0.110	0.310	-0.02116177	-1.03961727	0.01037052	1.03966899
0.150	0.310	-0.02308992	-1.03182937	0.01748594	1.03197751
0.190	0.310	-0.02193956	-1.02387067	0.02910578	1.02428428
0.230	0.310	-0.01770895	-1.01528662	0.05152088	1.01659299
0.270	0.310	-0.01039633	-1.00262343	0.11254379	1.00892013
0.307	0.313	-0.00196843	-0.90949464	0.42216929	1.00270006
0.030	0.270	-0.00692050	-1.04718486	0.00245159	1.04718773
0.070	0.270	-0.01345874	-1.03948153	0.00679358	1.03950372
0.110	0.270	-0.01692233	-1.03173263	0.01319399	1.03181698
0.150	0.270	-0.01730977	-1.02385556	0.02364408	1.02412852
0.190	0.270	-0.01461953	-1.01549485	0.04385416	1.01644133
0.230	0.270	-0.00885010	-1.00385655	0.09943140	1.00876884
0.267	0.273	-0.00171047	-0.91949526	0.39948646	1.00252729
0.030	0.230	-0.00576523	-1.03933267	0.00298813	1.03933696
0.070	0.230	-0.01076326	-1.03161792	0.00861383	1.03165388
0.110	0.230	-0.01268705	-1.02381455	0.01779183	1.02396912
0.150	0.230	-0.01153528	-1.01566193	0.03558312	1.01628506
0.190	0.230	-0.00730669	-1.00502172	0.08502685	1.00861202
0.227	0.233	-0.00145301	-0.93096692	0.37148217	1.00234644
0.030	0.190	-0.00461092	-1.03148251	0.00377684	1.03148941
0.070	0.190	-0.00807004	-1.02374182	0.01158694	1.02380739
0.110	0.190	-0.00845530	-1.01577325	0.02675307	1.01612548
0.150	0.190	-0.00576564	-1.00606358	0.06935426	1.00845125
0.187	0.193	-0.00119598	-0.94391469	0.33667415	1.00215987
0.030	0.150	-0.00345741	-1.02363201	0.00506896	1.02364456
0.070	0.150	-0.00537869	-1.01581479	0.01741257	1.01596401
0.110	0.150	-0.00422649	-1.00692324	0.05244603	1.00828814
0.147	0.153	-0.00093930	-0.95816901	0.29301283	1.00197024
0.030	0.110	-0.00230454	-1.01577337	0.00761441	1.01580189
0.070	0.110	-0.00268880	-1.00753898	0.03434670	1.00812424
0.107	0.113	-0.00068291	-0.97318004	0.23766570	1.00178061
0.030	0.070	-0.00115212	-1.00784771	0.01511684	1.00796106
0.067	0.073	-0.00042672	-0.98761289	0.16677153	1.00159470
0.027	0.033	-0.00017067	-0.99858245	0.07529678	1.00141724

COMPUTER PROGRAM C

```
      DIMENSION P(51,51),T(51,51),TDX(51,51),TDY(51,51)
      READ (5,100) E,H,N
      NN=N-1
      HH=H*H
      H2=HH/2.
      H6=H/6.
      DO 5 I=1,N
    5 P(I,N)=0.0
      Y=-H
      DO 7 J=1,N
      Y=Y+H
      X=-H
      DO 7 I=1,J
      X=X+H
      SG=-1.0
      WX=0.
      WY=0.
      WXX=0.
      WXY=0.
      DO 6 NI=1,9
      FN=NI-1
      GG=(FN+0.5)*3.14159265
      GGY=GG*Y
      GGX=GG*X
      SHY=0.5*(EXP(GGY)-EXP(-GGY))
      CHY=EXP(-GGY)+SHY
      ANB=0.5*(EXP(GG)+EXP(-GG))*(1.+2.*FN)
      ANC=ANB*(1.+2.*FN)
      SG=-1.*SG
      CS=COS(GGX)
      SN=SIN(GGX)
      WX=WX+SG*SHY*CS/ANC
      WY=WY+SG*CHY*SN/ANC
      WXX=WXX+SG*SHY*SN/ANB
    6 WXY=WXY+SG*CHY*CS/ANB
      CC=16./3.14159265/3.14159265
      CB=8./3.14159265
      EX=-CC*WX
      EY=2.*X-CC*WY
      EXX=CB*WXX
      EYY=-EXX
      EXY=-CB*WXY
      SI=SQRT(EX*EX+EY*EY)
      T(I,J)=1.+E*SI
      TDX(I,J)=E*(EX*EXX+EY*(EXY+2.))/SI
    7 TDY(I,J)=E*(EX*EXY+EY*EYY)/SI
      Y=1.0
      J=N+1
   10 J=J-1
      K=J-1
      X=-H
      DO 20 I=1,K
```

```
      X=X+H
      TA=SIN(P(I,J))/COS(P(I,J))
   20 P(I,J-1)=P(I,J)+TA*(P(I+1,J)-P(I,J))-H*(TDX(I,J)-TA*TDY(I,J))/T(I,
     1J)
      Y=Y-H
      IF (J.GT.2) GO TO 10
      TQ=HH*H/32.
      Y=0.
      DO 45 J=2,NN
      Y=Y+H
      X=0.
      I=1
      SX=T(I,J)*COS(P(I,J))
      SY=T(I,J)*SIN(P(I,J))
      TQ=TQ+(SX*Y+SY*H/4.)*H2
      IF (J.EQ.2) GO TO 42
      K=J-1
      DO 40 I=2,K
      X=X+H
      SX=T(I,J)*COS(P(I,J))
      SY=T(I,J)*SIN(P(I,J))
   40 TQ=TQ+(SX*Y+SY*X)*HH
   42 I=J
      X=X+H
      SX=T(I,J)*COS(P(I,J))
      SY=T(I,J)*SIN(P(I,J))
      SXU=SX*5./6.+T(I,J+1)*COS(P(I,J+1))/6.
   45 TQ=TQ+(SXU*(Y+H6)+SY*(X-H6))*H2
      J=N
      Y=1.0
      YY=1.-H/4.
      X=-H
      DO 50 I=1,NN
      X=X+H
      SX=T(I,J)
      SY=0.
      AA=H2
      IF (I.EQ.1) AA=H2/2.
   50 TQ=TQ+SX*AA*YY
      TQ=TQ+HH*YY/8.
      TQQ=TQ*8.
      WRITE (6,103) TQQ
   60 WRITE (6,104) TQ
   80 STOP
  100 FORMAT (F10.3,F9.3,I6)
  103 FORMAT (31HK          DIMENSIONLESS TORQUE =,F12.8)
  104 FORMAT (31H           ONE-EIGHTH OF IT     =,F12.8)
      END
```

DATA CARD -

```
   0.100    0.020    51
```